Granular Computing using Information Tables

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Abstract. A simple and more concrete granular computing model may be developed using the notion of information tables. In this framework, each object in a finite nonempty universe is described by a finite set of attributes. Based on attribute values of objects, one may decompose the universe into parts called granules. Objects in each granule share the same or similar description in terms of their attribute values. Studies along this line have been carried out in the theories of rough sets and databases. Within the proposed model, this paper reviews the pertinent existing results and presents their generalizations and applications.

1 Introduction

The concept of information granulation was first introduced by Zadeh [43] in the context of fuzzy sets in 1979. The basic ideas of crisp information granulation have appeared in related fields, such as interval analysis, quantization, rough set theory, Dempster-Shafer theory of belief functions, divide and conquer, cluster analysis, machine learning, databases, and many others. However, fuzzy information granulation has not received enough attention [47]. In a series of recent papers and invited talks, Zadeh [45–47,49] proposed the development of a theory of fuzzy information granulation. Motivated by the work of Zadeh, there is a fast growing interest in the study of information granulation and computations under the umbrella of *Granular Computing* (GrC)[1]. Roughly speaking, "GrC is a superset of the theory of fuzzy information granulation, rough set theory and interval computations, and is a subset of granular mathematics." [48]

There are theoretical and practical reasons for the study of granular computing. Many authors argued that information granulation is very essential

[1] The term "Granular Computing" was suggested by T.Y. Lin to label studies on information granulation and computations [45]. A Special Interest Group in Granular Computing in Berkeley Initiative in Soft Computing (BISC/SIG GrC) was established in 1997 (URL: http://www.mathcs.sjsu.edu/GrC/GrC.html). The coordinators of the group are Tsau Young Lin (leader), Frank Hoffmann, Yiyu Yao, and Ning Zhong.

to human problem solving, and hence has a very significant impact on the design and implementation of intelligent systems. Zadeh [47] identified three basic concepts that underlie human cognition, namely, granulation, organization, and causation. "Granulation involves decomposition of whole into parts, organization involves integration of parts into whole, and causation involves association of causes and effects." Yager and Filev [29] pointed out that "human beings have been developed a granular view of the world", and "...objects with which mankind perceives, measures, conceptualizes and reasons are granular". In many practical situations, when a problem involves incomplete, uncertain, or vague information, it may be difficult to differentiate distinct elements and one is forced to consider granules. A typical example is the theory of rough sets [20]. In some situations, although detailed information may be available, it may be sufficient to use granules in order to have an efficient and practical solution. Very precise solutions may in fact not be required for many practical problems. It may also happen that the acquisition of precise information is too costly, and coarse-grained information reduces cost.

In summary, granular computing is inspired by the ways in which humans granulate information and reason with coarse-grained information. It builds on existing machinery for fuzzy information processing, such as linguistic variables, fuzzy if-then rules and fuzzy graphs, generalized constraints, and computing with words [47]. Granular computing is likely to play an important role in the evolution of fuzzy logic and its applications.

There are at least three fundamental issues in granular computing: granulation of the universe, description of granules, and relationships between granules. Granulation involves decomposition of whole into parts. A granule is "a clump of points (objects) drawn together by indistinguishability, similarity, proximity or functionality" [47]. In order to apply this abstract concept, it is necessary to study criteria for deciding if two elements should be put into the same granule. In other words, one must provide necessary semantics interpretations for notions such as indistinguishability, similarity, and proximity. Two structures can be observed from the granulation of a universe, the structure of each individual granules and structure induced by a family of granules. In general, a larger granule may be further divided into smaller granules, while smaller granules may be combined into a larger granules. In this way, one may obtain stratified granulation structures of a universe [34]. Once constructed, it is necessary to describe, to name and to label granules using certain languages. Each label represents a concept such that an element in the granule is an instance of the named category, as being done in classification [7]. The granulated view summarizes available information and knowledge about the universe. By considering a class of objects sharing similar properties, instead of individuals, one may be able to establish relationships and connection between granules. In fact, this is one of the main tasks of data mining [42]. It may be argued that the construction, interpretation, description, and connections of granules are of fundamental importance in the understanding, representation, organization and synthesis of data, information, and knowledge.

A systematic study and a general framework of granular computing were given in a recent paper by Zadeh [47]. Granules are constructed and defined based on the concept of generalized constraints. Examples of constraints are equality, possibilistic, probabilistic, fuzzy, and veristic constraints. Granules are labeled by fuzzy sets or natural language words. Relationships between granules are represented in terms of fuzzy graphs or fuzzy if-then rules. The associated computation method is known as computing with words [44]. On the other hand, many researchers investigated specific and more concrete models of granular computing. Lin [9] and Yao [33] studied granular computing using neighborhood systems for the interpretation of granules. Pawlak [20], Skowron and Stepaniuk [24], and Polkowski and Skowron [21] examined granular computing in connection with the theory of rough sets. A salient features of these studies is that a particular semantics interpretation of granules is defined, and an algorithm for constructing granules is given.

The main objective of the present study is to develop a simple and more concrete model for non-fuzzy granular computing using information tables. With respect to the proposed model, we review studies on non-fuzzy granular computing and investigate their possible generalizations and applications. In this framework, each object in a finite nonempty universe is described by a finite set of attributes. That is, each object is only perceived, observed, or measured by using a finite number of properties. The universe is decomposed into granules by grouping objects with the same or similar properties. The representation of objects by their attribute values provide the semantics for interpreting the induced granules. For example, a patient may be represented by a set of symptoms. A set of patients may be divided into subgroups such that each subgroup of patients suffer from the same disease characterized by certain symptoms. Several types of relationships between attribute values will be considered. They induce different granulation structures on the universe.

To illustrate the usefulness of the proposed framework, at the end of this paper we also discuss two specific problems of granular computing, namely, approximations induced by granulations and relationships between granules in data analysis and mining.

2 Information Tables and a Decision Logic Language

The notion of information tables has been studied by many authors as a simple knowledge representation method, in which objects are described by their values on a set of attributes [1,12,15,17,19,27,37]. Formally, an information table is a quadruple:

$$S = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}),$$

where

$$U \text{ is a finite nonempty set of objects,}$$
$$At \text{ is a finite nonempty set of attributes,}$$
$$V_a \text{ is a nonempty set of values for } a \in At,$$
$$I_a : U \longrightarrow V_a \text{ is an information function.}$$

Each information function I_a is a total function that maps an object of U to exactly one value in V_a. Similar representation schemes may be found in many fields, such as decision theory, pattern recognition, machine learning, data analysis, data mining, and cluster analysis [19].

An information table can be conveniently presented in a table form. Table 1, taken from an example in [22], is an example of an information table. The columns are labeled by attributes, the rows are labeled by objects, and each row represents the information about that object. Pawlak [19] referred to an information table as an information system, a knowledge representation system, or an attribute-value system. We prefer to use the name information tables, in order to avoid confusion with the commonly associated meaning of information systems [9].

Object	Height	Hair	Eyes	Class
o_1	short	blond	blue	+
o_2	short	blond	brown	-
o_3	tall	red	blue	+
o_4	tall	dark	blue	-
o_5	tall	dark	blue	-
o_6	tall	blond	blue	+
o_7	tall	dark	brown	-
o_8	short	blond	brown	-

Table 1. An information table

An information table contains all available information about the objects in the universe. Objects are perceived and observed only through their properties. Objects with the same description cannot be distinguished and they are considered to be the same [20]. More generally, objects with similar descriptions may also be considered to be approximately the same. This leads to granulations of the universe. Granular computing using information table deals mainly with the decomposition of universe based on objects' descriptions. Information table is a more concrete model that provides semantics for the notion of granules. However, it is only one of the possible ways in which granules are formed and interpreted.

Information provided by an information table may also be described in terms of certain logic languages, in order to make inference easily. Pawlak [19] discussed a decision logic language (DL-language) with respect to information tables. It is a language for describing objects or a group of objects of the universe. For example, an object can be represented as a conjunction of attribute-value pairs. A subset of objects can be similarly described. Formally, an atomic formula in the DL-language is given by (a, v), where $a \in At$ and $v \in V_a$. If ϕ and ψ are formulas in the DL-language, then so are $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, and $\phi \equiv \psi$. The semantics of the DL-language is defined in Tarski's style through the notions of a model and satisfiability. The model is an information table S, which provides interpretation for symbols and formulas of the DL-language. The satisfiability of a formula ϕ by an object x, written $x \models_S \phi$ or in short $x \models \phi$ if S is understood, is interpreted as follows:

(1) $x \models (a, v)$ iff $I_a(x) = v$,
(2) $x \models \neg\phi$ iff not $x \models \phi$,
(3) $x \models \phi \wedge \psi$ iff $x \models \phi$ and $x \models \psi$,
(4) $x \models \phi \vee \psi$ iff $x \models \phi$ or $x \models \psi$,
(5) $x \models \phi \rightarrow \psi$ iff $x \models \neg\phi \vee \psi$,
(6) $x \models \phi \equiv \psi$ iff $x \models \phi \rightarrow \psi$ and $x \models \psi \rightarrow \phi$.

The first four formulas are in fact used in the evaluation of satisfiability of queries by objects in database systems. For a formula ϕ, the set of objects satisfying ϕ is given by:

$$m_S(\phi) = \{x \in U \mid x \models \phi\}. \tag{1}$$

It is called the meaning of the formula ϕ in S. If S is understood, we simply write $m(\phi)$. The meaning of a formula ϕ is the set of all objects having the property expressed by the formula ϕ. Therefore, ϕ may be viewed as a description of the set of objects $m(\phi)$. Two distinct formulas may have the same meaning in an information table. A granule may have different representations. The connections between formulas of the DL-language and subsets of U are expressed as [19]:

(a) $m(a, v) = \{x \in U \mid I_a(x) = v\}$,
(b) $m(\neg\phi) = -m(\phi)$,
(c) $m(\phi \wedge \psi) = m(\phi) \cap m(\psi)$,
(d) $m(\phi \vee \psi) = m(\phi) \cup m(\psi)$,
(e) $m(\phi \rightarrow \psi) = -m(\phi) \cup m(\psi)$,
(f) $m(\phi \equiv \psi) = (m(\phi) \cap m(\psi)) \cup (-m(\phi) \cap -m(\psi))$,

where $-m(\phi) = U - m(\phi)$ denotes the set complement of $m(\phi)$. They give a set-theoretic interpretation of logic operations. In particular, logic negation, conjunction, and disjunction are interpreted as set complement, intersection, and union, respectively.

A formula ϕ is said to be true in an information table S, written $\models_S \phi$, if and only if ϕ is satisfied by all objects in the universe, namely, $m(\phi) = U$. It is false if and only if no object satisfies the formula, namely, $m(\phi) = \emptyset$. For two formulas ϕ and ψ, $\phi \rightarrow \psi$ is true if and only if every object satisfying ϕ also satisfies ψ, namely, $m(\phi) \subseteq m(\psi)$. They are equivalent in S if and only if $m(\phi) = m(\psi)$. In summary, we have [19]:

$$
\begin{aligned}
&\text{(i)} && \models_S \phi \text{ iff } m(\phi) = U,\\
&\text{(ii)} && \models_S \neg\phi \text{ iff } m(\phi) = \emptyset,\\
&\text{(iii)} && \models_S \phi \rightarrow \psi \text{ iff } m(\phi) \subseteq m(\psi),\\
&\text{(iv)} && \models_S \phi \equiv \psi \text{ iff } m(\phi) = m(\psi).
\end{aligned}
$$

We can therefore study the relationships between concepts described by formulas of the *DL*-language based on the relationships between their corresponding sets of objects.

For the information table 1, the following expressions are examples of formulas of the *DL*-language:

$$
\begin{aligned}
&(\text{Height},\ \text{tall}),\\
&(\text{Height},\ \text{short}),\\
&(\text{Hair},\ \text{dark}),\\
&(\text{Height},\ \text{tall}) \vee (\text{Height},\ \text{short}),\\
&(\text{Height},\ \text{tall}) \wedge (\text{Hair},\ \text{dark}),\\
&(\text{Height},\ \text{tall}) \vee (\text{Hair},\ \text{dark}),\\
&(\text{Hair},\ \text{dark}) \rightarrow (\text{Height},\ \text{tall}),\\
&(\text{Hair},\ \text{dark}) \equiv (\text{Height},\ \text{tall}).
\end{aligned}
$$

The meanings of these formulas, i.e., the subsets of objects satisfying the formulas, are given by:

$$
\begin{aligned}
&m(\text{Height},\ \text{tall}) = \{o_3, o_4, o_5, o_6, o_7\},\\
&m(\text{Height},\ \text{short}) = \{o_1, o_2, o_8\},\\
&m((\text{Height},\ \text{tall}) \vee (\text{Height},\ \text{short})) = U,\\
&m(\text{Hair},\ \text{dark}) = \{o_4, o_5, o_7\},\\
&m((\text{Height},\ \text{tall}) \wedge (\text{Hair},\ \text{dark})) = \{o_4, o_5, o_7\},\\
&m((\text{Height},\ \text{tall}) \vee (\text{Hair},\ \text{dark})) = \{o_3, o_4, o_5, o_6, o_7\},\\
&m((\text{Hair},\ \text{dark}) \rightarrow (\text{Height},\ \text{tall})) = U,\\
&m((\text{Hair},\ \text{dark}) \equiv (\text{Height},\ \text{tall})) = \{o_1, o_2, o_4, o_5, o_7, o_8\}.
\end{aligned}
$$

Among these formulas, two are true in the information table, namely:

$$
\begin{aligned}
&\models_S (\text{Height},\ \text{tall}) \vee (\text{Height},\ \text{short}),\\
&\models_S (\text{Hair},\ \text{dark}) \rightarrow (\text{Height},\ \text{tall}).
\end{aligned}
$$

The first formula represents the fact that an object's Height is either tall or short. The second formula represents the fact that if an object's Hair is dark, then its Height is tall. For the subset $\{o_4, o_5, o_7\}$, it can be described by both formulas (Hair, dark) and (Height, tall) $\wedge$ (Hair, dark). This suggests that $\models_S$ (Hair, dark) $\equiv$ ((Height, tall) $\wedge$ (Hair, dark)).

The decision logic language, *DL*-Language, has been studied by many authors. Orlowska [16] used a similar logic for studying reasoning with vague concepts. Polkowski and Skowron [21] adopted decision logic for the formulation of an adaptive calculus of granules in the context of information tables.

3 Construction and Interpretation of Granules in Information Tables

In the order of generality, this section summarizes the constructions of granules using the equality relation, equivalence relations, and reflexive binary relations on attribute values. The common practice of using the equality relation, as being done in the development of rough set theory [18], is based on exact value matching. This in fact does not take into too much consideration of semantic relationships between attribute values. By using other types of relations, semantic relationships between attribute values can be easily integrated into information tables.

3.1 Granules induced by equality of attribute values

With respect to an attribute $a \in At$, two objects o and o' may have the same value, namely, $I_a(o) = I_a(o')$. In this case, one cannot differentiate o from o' based solely on their values on attribute a. They may be put into the same granule. For $v \in V_a$, one obtains the granule corresponding to the atomic formula (a, v):

$$\begin{aligned} G_e(a, v) &= \{x \in U \mid I_a(x) = v\} \\ &= m(a, v). \end{aligned} \tag{2}$$

This granule consists of all objects whose value on attribute a is *equal* to v. Such granules are defined by equality constraints in the sense discussed by Zadeh [47].

The family of granules,

$$\pi_{\{a\}} = \{G_e(a, v) \neq \emptyset \mid v \in V_a\}, \tag{3}$$

forms a partition of the universe [19]. The corresponding equivalence relation $EQ_{\{a\}}$ on U is given by:

$$oEQ_{\{a\}}o' \Longleftrightarrow I_a(o) = I_a(o'). \tag{4}$$

Each equivalence class of the relation $EQ_{\{a\}}$ is a granule. The equivalence class containing $o \in U$, written $[o]_{EQ_{\{a\}}}$, is defined by collecting all objects whose value on attribute a is the same as o's value:

$$\begin{aligned} [o]_{EQ_{\{a\}}} &= \{x \in U \mid I_a(x) = I_a(o)\} \\ &= G_e(a, I_a(o)). \end{aligned} \tag{5}$$

The partition $\pi_{\{a\}}$ of the universe is referred to as a quotient set of U and is denoted by $U/EQ_{\{a\}}$. It offers a granulated view of the universe. The sets in $\pi_{\{a\}}$ are called elementary granules, as they are the smallest granules derivable based on values of attribute a. From the elementary granules, large granules may be built by taking a union of a family of elementary granules. One can build a hierarchy of granules. If the empty set $\emptyset$ is added, one obtains a sub-Boolean algebra of the Boolean algebra 2^U formed by the power set of U.

The argument for constructing granules can be easily extended to cases of more than one attribute. For a pair of attributes $a, b \in At$ and two values $v_a \in V_a, v_b \in V_b$, one can obtain the following granule corresponding to the formula $(a, v_a) \wedge (b, v_b)$:

$$\begin{aligned} G_e((a, v_a) \wedge (b, v_b)) &= \{x \in U \mid I_a(x) = v_a \wedge I_b(x) = v_b\} \\ &= m((a, v_a) \wedge (b, v_b)) \\ &= m(a, v_a) \cap m(b, v_b) \\ &= G_e(a, v_a) \cap G_e(b, v_b). \end{aligned} \tag{6}$$

The granule is defined by two equality constraints. The family of granules:

$$\pi_{\{a,b\}} = \{G_e((a, v_a) \wedge (b, v_b)) \neq \emptyset \mid v_a \in V_a, v_b \in V_b\}, \tag{7}$$

is a partition of the universe. The corresponding equivalence relation is given by $EQ_{\{a,b\}} = EQ_{\{a\}} \cap EQ_{\{b\}}$, namely,

$$oEQ_{\{a,b\}}o' \Longleftrightarrow I_a(o) = I_a(o') \wedge I_b(o) = I_b(o'). \tag{8}$$

Granules in the partition $\pi_{\{a,b\}}$ are smaller than granules in partitions $\pi_{\{a\}}$ and $\pi_{\{b\}}$.

For information table 1, with respect to the attribute $A = \{\text{Hair}\}$, we can partition the universe into equivalence classes:

$$\{o_1, o_2, o_6, o_8\}, \quad \{o_3\}, \quad \{o_4, o_5, o_7\}.$$

They correspond to formulas (Hair, blond), (Hair, red), and (Hair, dark). Similarly, the use of attribute Height produces the partition:

$$\{o_1, o_2, o_8\}, \quad \{o_3, o_4, o_5, o_6, o_7\}.$$

When the pair of attributes $A = \{\text{Height}, \text{Hair}\}$ is used, we consider all possible combinations of values of Height and Hair, such as $(\text{Height}, \text{ short}) \wedge$

(Hair, blond), (Height, tall) $\wedge$ (Hair, blond), and so on. They produce the partition of the universe:

$$\{o_1, o_2, o_8\}, \quad \{o_3\}, \quad \{o_4, o_5, o_7\}, \quad \{o_6\}.$$

They are granules finer than the ones produced by using either Height or Hair.

For a subset of attributes $A \subseteq At$, the equivalence relation is given by $EQ_A = \bigcap_{a \in A} EQ_{\{a\}}$, each equivalence class (granule) is defined by the equality constraints $\bigwedge_{a \in A} I_a(x) = v_a$, where $v_a \in V_a$. The algebra $(\{EQ_A\}_{A \subseteq At}, \cap)$ is a lower semi-lattice with the zero element EQ_{At} [15]. For two subsets of attributes $A, B \subseteq At$, if $EQ_A \subset EQ_B$, we say that the partition π_A is finer than π_B, or π_B is coarser than π_A. We will also say that π_A is a specialization, or refinement, of π_B, or π_B is a generalization, or coarsening, of π_A [19]. The order relation of the semi-lattice represents the generalization-specialization relationships between partitions, i.e., families of elementary granules. The empty set $\emptyset$ produces the coarsest equivalence relation, i.e., $EQ_\emptyset = U \times U$, where $\times$ denotes the Cartesian product of sets. The entire set of attributes produces the finest equivalence relation EQ_{At}. In the construction of granules, the addition of an attribute leads to a specialization, and hence smaller elementary granules. Conversely, the deletion of an attribute leads to a generalization, and hence larger elementary granules.

One may study relationships between attributes using partitions induced by individual attributes or subsets of attributes [39]. This can be done in an information-theoretic setting, as discussed by Lee [8] and Malvestuto [14] on the issues of correlation and interdependency among attributes. The notions such as functional, multi-valued, hierarchical and join dependencies are stated in terms of various entropy functions. Additional entropy related measures and their uses in machine learning and data mining can be found in a paper by Yao *et al.* [39].

3.2 Granules induced by equivalence of attribute values

Granules constructed using a single attribute may be either too large or too small. The addition of more attributes may resolve the former problem. A solution to the latter problem will be discussed in this section by grouping values in V_a. In particular, values in V_a are divided into disjoint classes, i.e., a partition of V_a, and the corresponding equivalence classes are used as new attribute values. Two examples of such approaches are the discretization of real-valued attributes and the use of concept hierarchies [5]. The idea can be formalized by introducing equivalence relations on the set of attribute values [4,28].

Suppose E_a is an equivalence relation on the set of values V_a of an attribute $a \in At$. It partitions the set V_a into a disjoint family of subsets V_a/E_a called quotient set of V_a. Let $[v]_{E_a}$ denote the equivalence class containing v.

For $v \in V_a$, we obtain a granule by replacing $=$ with E_a in Equation (2):

$$\begin{aligned} G_E(a, v) &= \{x \in U \mid I_a(x) E_a v\} \\ &= \{x \in U \mid I_a(x) \in [v]_{E_a}\} \\ &= \bigcup \{m(a, v') \mid v' \in V_a, v' \in [v]_{E_a}\} \\ &= \bigcup \{m(a, v') \mid v' \in V_a, v' E_a v\}. \end{aligned} \tag{9}$$

It consists of all objects whose value on attribute a is *equivalent* to v. The equivalence relation is a generalization of the trivial equality relation $=$. The granules given by Equation (9) may be interpreted as the granule defined by a generalized equality constraint. Many authors [5,9–11] used the equivalence classes $[v]_{E_a}$ as higher level concepts in a concept hierarchy. Each value in $v \in V_a$ is replaced by its equivalence class $[v]_{E_a}$ in the original information table to produce a quotient information table. For the quotient information table, the following equality constraint can in fact be used: for $[v]_{E_a} \in V_a/E_a$,

$$\begin{aligned} G_e(a', [v]_{E_a}) &= \{x \in U \mid [I_a(x)]_{E_a} = [v]_{E_a}\} \\ &= m(a', [v]_{E_a}), \end{aligned} \tag{10}$$

where a' is used to explicitly express the fact that in the quotient information table, an attribute takes equivalence classes of V_a as its values. For two values $v E_a v'$, $m(a', [v]_{E_a}) = m(a', [v']_{E_a})$.

The family of granules,

$$\Pi_{\{a\}} = \{G_E(a, v) \neq \emptyset \mid v \in V_a\}, \tag{11}$$

form a partition of the universe [19]. The corresponding equivalence relation $E_{\{a\}}$ on U is given by:

$$o E_{\{a\}} o' \Longleftrightarrow I_a(o) E_a I_a(o'). \tag{12}$$

The equivalence class containing $o \in U$, written $[o]_{E_{\{a\}}}$, is:

$$\begin{aligned} [o]_{E_{\{a\}}} &= \{x \in U \mid I_a(x) E_a I_a(o)\} \\ &= G_E(a, I_a(o)) \\ &= \bigcup \{m(a, I(o')) \mid o' \in U, I_a(o') E_a I_a(o)\} \\ &= \bigcup \{[o']_{EQ_{\{a\}}} \mid o' \in U, I_a(o') E_a I_a(o)\}. \end{aligned} \tag{13}$$

It consists of all objects whose value on attribute a is equivalent to that of the object o. Each of such equivalence granules is a union of some smaller granules of the equivalence relation defined by the equality relation. The argument can be easily extended to any subset of attributes.

Equivalence classes in V_a/E_a can be combined again to form even larger granules. The process can be continued until the right sized granules are obtained. Alternatively, one may use a sequence of nested equivalence relations on attribute values. This leads to the formation of a concept hierarchy. Each equivalence class of attribute values is viewed as a concept. A finer equivalence relation produces more specific concepts, while a coarser relation produces more general concepts [34].

3.3 Granules induced by similarity of attribute values

The use of the trivial equality relation $=$ and equivalence relations on attribute values provides a straightforward way for defining granulation structures on the universe. The type of granulation structures is characterized by partitions of the universe. With a fixed information table, from a subset of attributes one can obtain a partition. The converse is not necessarily true. For an arbitrary partition, one may not be able to find a subset of the attributes producing the same partition. Furthermore, equality and equivalence represent special cases of similarity. In order to obtain additional granulation structures, one may use other types of similarity relation on the attribute values [38,40].

Suppose R_a is a binary relation on V_a. For $v, v' \in V_a$, if vR_av' we say that v' is R_a-related to v, v is a predecessor of v', and v' is a successor of v. The binary relation R_a is interpreted as defining the similarity of attribute values. A value v is similar to v' if vR_av'. It seems reasonable to assume that R_a is reflexive, i.e., a value is similar to itself. The property of symmetry may not necessarily be required, namely, the similarity may not be symmetric [9,25]. By collecting values similar to v, we can form a granule of V_a as follows:

$$R_a^p(v) = \{v' \mid v' \in V_a, v'R_av\}. \tag{14}$$

The set $R_a^p(v)$ is called the predecessor neighborhood of v induced by the binary relation [31]. A binary relation and the predecessor neighborhoods uniquely determine each other. By the reflexivity of R_a, the family of granules $\{R_a^p(v) \neq \emptyset \mid v \in V_a\}$ forms a covering of V_a, which is not necessarily a partition.

For $v \in V_a$, by extending Equation (9), we obtain a granule of U:

$$\begin{aligned} G_s(a, v) &= \{x \in U \mid I_a(x)R_av\} \\ &= \{x \in U \mid I_a(x) \in R_a^p(v)\} \\ &= \bigcup\{m(a, v') \mid v' \in V_a, v' \in R_a^p(v)\}. \end{aligned} \tag{15}$$

It consists of all objects whose value on attribute a is *similar* to v. The family of granules,

$$C_{\{a\}} = \{G_s(a, v) \neq \emptyset \mid v \in V_a\}, \tag{16}$$

form a covering of the universe. Each granule is in fact the predecessor neighborhood of certain element of universe induced by the binary relation $R_{\{a\}}$ on U:

$$oR_{\{a\}}o' \Longleftrightarrow I_a(o)R_aI_a(o'). \tag{17}$$

Unlike the cases of equality and equivalence where an element of the universe belongs to exactly one equivalence class, the element may belong to more than one granule. In fact, o is a member of each of the following family of granules,

$$\{G_s(a, v) \mid I_a(o) \in R_a^p(v), v \in V_a\}. \tag{18}$$

One the other hand, the granule:

$$\begin{aligned} R^p_{\{a\}}(o) &= \{x \in U \mid I_a(x) R_a I_a(o)\} \\ &= G_s(a, I_a(o)), \end{aligned} \tag{19}$$

consists of those elements whose value on a is similar to o's value. The relation $R_{\{a\}}$ preserves properties of R_a. For example, if R_a is a reflexive, a symmetric, and a transitive relation, $R_{\{a\}}$ is a reflexive, a symmetric, and a transitive relation, respectively [40].

For a subset of attributes $A \subseteq At$, the similarity constraint is given by $\bigwedge_{a \in A} I_a(x) R_a v_a$, where $v_a \in V_a$. The similarity relation on the universe defined by A is given by:

$$\begin{aligned} o R_A o' &\Longleftrightarrow \textstyle\bigwedge_{a \in A} \ I_a(o) R_a I_a(o') \\ &\Longleftrightarrow \textstyle\bigwedge_{a \in A} \ o R_{\{a\}} o'. \end{aligned} \tag{20}$$

That is, $R_A = \bigcap_{a \in A} R_{\{a\}}$. The relation R_A only preserves the common properties of relations $R_{\{a\}}$'s, $a \in A$. Similarly, an element may belong to more than one granule.

The same process may also used to construct granules using other neighborhoods of a similarity relation. A detailed discussion can be found in a paper by Yao [31]. Furthermore, all concepts and observations discussed in the last section, such as generalization, specialization, and semi-lattice structure of granulations, may be examined for the cases of arbitrary similarity relations.

4 Rough Set Approximations

With the granulation of a universe, an arbitrary subset of the universe cannot be represented precisely using granules. One needs to deal with its approximations. This is in fact one of the main issues of the theory of rough sets [19]. Our discussion of this section follows, to a large extent, a paper by Yao [35].

Consider an equivalence relation $E \subseteq U \times U$ on a universe U. The pair $apr = (U, E)$ is called an approximation space. With respect to the partition U/E, an arbitrary set $X \subseteq U$ may not necessarily be a union of some equivalence classes. One may characterize X by a pair of lower and upper approximations:

$$\begin{aligned} \underline{apr}(X) &= \bigcup \{G \mid G \in U/E, G \subseteq X\}, \\ \overline{apr}(X) &= \bigcup \{G \mid G \in U/E, G \cap X \neq \emptyset\}. \end{aligned} \tag{21}$$

The lower approximation $\underline{apr}(X)$ is the union of all the equivalence granules which are subsets of X. The upper approximation $\overline{apr}(X)$ is the union of all the equivalence granules which have a non-empty intersection with X.

Lower and upper approximations are dual to each other in the sense:

$$\begin{aligned} &\text{(Ia)} \quad \underline{apr}(X) = -\overline{apr}(-X), \\ &\text{(Ib)} \quad \overline{apr}(X) = -\underline{apr}(-X). \end{aligned}$$

The set X lies within its lower and upper approximations:

$$\text{(II)} \qquad \underline{apr}(X) \subseteq X \subseteq \overline{apr}(X).$$

Intuitively, lower approximation may be understood as the pessimistic view and the upper approximation the optimistic view in approximating a set by using equivalence granules. One can also verify the following properties:

$$\begin{aligned} &\text{(IIIa)} \quad \underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y), \\ &\text{(IIIb)} \quad \overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y). \end{aligned}$$

The lower (upper) approximation of the intersection (union) of a finite number of sets can be obtained from their lower (upper) approximations. However, we only have:

$$\begin{aligned} &\text{(IVa)} \quad \underline{apr}(X \cup Y) \supseteq \underline{apr}(X) \cup \underline{apr}(Y), \\ &\text{(IVb)} \quad \overline{apr}(X \cap Y) \subseteq \overline{apr}(X) \cap \overline{apr}(Y). \end{aligned}$$

It is impossible to obtain the lower (upper) approximation of the union (intersection) of some sets from their lower (upper) approximations. Additional properties of rough set approximations can be found in papers by Pawlak [18], and Yao [32,36].

Equivalence classes of the partition U/E are called the elementary granules. They represent the available information. All knowledge we have about the universe are about these elementary granules, instead of about individual elements. It follows that we also have knowledge about the union of some elementary granules. As a matter of fact, if X is the empty set $\emptyset$ or the union of one or more elementary granules, then $\underline{apr}(X) = X = \overline{apr}(X)$. These sets are called definable, observable, measurable, or composed granules. The set of all definable granules is denoted $GK(U)$, which is a subset of the power set 2^U. The set $GK(U)$ is closed under both set intersection and union. It is an σ-algebra of subsets of U generated by the family of equivalence classes U/E.

For an element $G \in GK(U)$, we have:

$$\underline{apr}(G) = G = \overline{apr}(G). \tag{22}$$

For any subset $X \subseteq U$, we have the equivalent definition of rough set approximations:

$$\underline{apr}(X) = \bigcup \{G \mid G \in GK(U), G \subseteq X\},$$
$$\overline{apr}(X) = \bigcap \{G \mid G \in GK(U), X \subseteq G\}. \tag{23}$$

This definition offers another interesting interpretation. The lower approximation is the largest definable granule contained in X, where the upper approximation is the smallest definable granule containing X. They therefore represent the best approximations of X from below and above using definable granules.

From similarity relations on attribute values, one can derive a similarity relation R on the universe U. A covering of the universe can be constructed by using a particular type of neighborhoods of all elements of U. Let U/R denote such a covering. Rough set approximations can be defined by generalizing Equation (21). In particular, an equivalence class is replaced by a granule in U/R. One of such generalizations is [33]:

$$\underline{apr}(X) = \bigcup \{G \mid G \in U/R, G \subseteq X\},$$
$$\overline{apr}(X) = -\underline{apr}(-X). \tag{24}$$

In this definition, we generalize the lower approximation and define the upper approximation through duality. In general, $\overline{apr}(X)$ is different from the straightforward generalization $\bigcup \{G \mid G \in U/R, G \cap X \neq \emptyset\}$. While the lower approximation is the union of some granules, the upper approximation cannot be expressed in this way [33].

Subsets in the covering U/R are called elementary granules. By definition, if X is a union of some elementary granules in U/R, then we have $\underline{apr}(X) = X$. That is, X can be defined by granules in U/R exactly from below. For this reason, the empty set $\emptyset$ or the union of some elementary granules are referred to as lower definable granules. The set of all lower definable granules $\underline{GK}(U)$ is the minimum subset of 2^U that contains $\emptyset$ and U/R, and is closed under set union. The complemented system:

$$\overline{GK}(U) = \{-G \mid G \in \underline{GK}(U)\}, \tag{25}$$

contains U and is closed under set intersection. In other words, $\overline{GK}(U)$ is a closure system [31]. For an element $G \in \underline{GK}(U)$, i.e., $-G \in \overline{GK}(U)$, we have:

$$\underline{apr}(G) = G,$$
$$\overline{apr}(-G) = -G. \tag{26}$$

That is, the system $\overline{GK}(U)$ consists of upper definable subsets of U. In general, $G = \underline{apr}(G) \neq \overline{apr}(G)$ and $\underline{apr}(G') \neq \overline{apr}(G') = G'$ for elements

$G \in \underline{GK}(U)$ and $G' \in \overline{GK}(U)$. In terms of lower and upper definable granules, we have another equivalent definition:

$$\underline{apr}(X) = \bigcup\{G \mid G \in \underline{GK}(U), G \subseteq X\},$$
$$\overline{apr}(X) = \bigcap\{G' \mid G' \in \overline{GK}(U), X \subseteq G'\}. \tag{27}$$

The lower approximation is the largest lower definable granule contained in X, and the upper approximation is the smallest upper definable granules containing X. They are related to the definition for the case of partitions, in which $\underline{GK}(U)$ and $\overline{GK}(U)$ become the same. For a covering, the set $\underline{GK}(U) \cap \overline{GK}(U)$ consists of both lower and upper definable granules. Obviously, $\emptyset, U \in \underline{GK}(U) \cap \overline{GK}(U)$.

The new approximations satisfy properties (I), (II), and (IV). They do not satisfy property (III). Nevertheless, they satisfy a weaker version:

$$\text{(Va)} \quad \underline{apr}(X \cap Y) \subseteq \underline{apr}(X) \cap \underline{apr}(Y),$$
$$\text{(Vb)} \quad \overline{apr}(X \cup Y) \supseteq \overline{apr}(X) \cup \overline{apr}(Y).$$

By definition, $\underline{apr}(X \cap Y)$ can be written as a union of some elementary granules. Although both $\underline{apr}(X)$ and $\underline{apr}(Y)$ can be expressed as unions of elementary granules, $\underline{apr}(X) \cap \underline{apr}(Y)$ cannot be so expressed.

5 Data Analysis and Data Mining

A granule represents a concept such that each element in the granule is an instance of the concept. Under this interpretation, one of the tasks of data analysis, knowledge discovery and data mining may be regarded to as finding connections between concepts represented by their corresponding granules. In the framework of granular computing, the main results from Yao and Zhong [41,42] are reviewed.

Let $X \subseteq U$ be a subset of the universe representing a certain concept ϕ_X, and FG a family of granules whose descriptions are known. We consider the task of finding a description of A in terms of granules in FG. For a granule $G \in FG$ with description ϕ_G, i.e., $m(\phi_G) = G$, we have either $G \cap X = \emptyset$ or $G \cap X \neq \emptyset$. For the case $G \cap X = \emptyset$, we say that G and X are not *positively* related. However, we have:

$$G \subseteq -X. \tag{28}$$

By property (iii), we have:

$$\models_S \phi_G \rightarrow \neg\phi_X. \tag{29}$$

Hence, we can establish an if-then type rule:

$$\text{IF} \quad \phi_G \quad \text{THEN} \quad \text{not } \phi_X. \tag{30}$$

This rule enables us to decide that an instance of G is not an instance of X. It gives the properties that make an element of U not to be an instance of X. For the case $G \cap X \neq \emptyset$, we consider three special sub-cases:

$$\begin{aligned} &\text{(a)}\ \ G \subseteq X, \\ &\text{(b)}\ \ G \supseteq X, \\ &\text{(c)}\ \ G = X. \end{aligned}$$

In decision logic languages, we have:

$$\begin{aligned} &\models_S \phi_G \rightarrow \phi_X, \\ &\models_S \phi_X \rightarrow \phi_G, \\ &\models_S \phi_G \equiv \phi_X. \end{aligned} \tag{31}$$

By properties (iii) and (iv), we can form the following set of rules:

$$\begin{aligned} \text{IF}\ \ &\phi_G\ \ \text{THEN}\ \ \phi_X, \\ \text{OIF}\ \ &\phi_G\ \ \text{THEN}\ \ \phi_X, \\ \text{IIF}\ \ &\phi_G\ \ \text{THEN}\ \ \phi_X, \end{aligned} \tag{32}$$

where OIF stands for "only if" and IIF stands for "if and only if". We express these rules slightly different from the conventional way, in order to see the difference between them. The first rule enables us to decide if an element of the universe is an instance of A. It shows the properties that make an element of U to be an instance of A. The second rule, which is normally expression as:

$$\text{IF}\ \ \phi_X\ \ \text{THEN}\ \ \phi_G, \tag{33}$$

tells us the properties that an instance of X must have. The third rule is the combination of the first two rules. It summarizes the properties that instances of X, and only instances of X, must have. The first two rules may be interpreted as one-way implication, and the third rule as two-way implication. In knowledge discovery and data mining, one may be interested in different rules depending on the situation. Typically, the first rule is referred to as a *decision* rule, while the second rule as a *characteristic* rule.

The rules obtained for the previous cases are certain rules, which reflect the logical relationships between concepts or granules. In some situations, though a strict logical connection does not exist, there may still exist some connection between two granules. This corresponds to the case where $G \cap X \neq \emptyset$ and neither $G \subseteq X$ nor $G \supseteq X$ is true. In order to characterize such associations between two concepts ϕ and ψ, one may generalize logical rules to association rules of the following form:

$$\text{IF}\ \ \phi\ \ \text{THEN}\ \ \psi \quad \text{with}\ \ \alpha_1, \ldots, \alpha_m, \tag{34}$$

where $\alpha_1, \ldots, \alpha_m$ denote the degree or strength of relationships [51]. Although keywords such as IF and THEN are used, one should not interpret the rules as expressing logical implications. Instead, these keywords are used to simply link concepts together [47]. For clarity, we also simply write $\phi \Rightarrow \psi$. The values $\alpha_1, \ldots, \alpha_m$ quantifies different types of uncertainty and properties associated with the rule. Examples of quantitative measures include confidence, uncertainty, applicability, quality, accuracy, and interestingness of rules. A recent systematic study on uncertain rules was given by Yao and Zhong [41].

Using the cardinalities of sets, we obtain the contingency Table 2, representing the quantitative information about the rule $\phi \Rightarrow \psi$, where $|\cdot|$ denotes the cardinality of a set. The values in the four cells are not independent. They are linked by the constraint $a + b + c + d = n$. The 2×2 contingency table has been used by many authors for representing information of rules [2,6,23,26,50].

	ψ	$\neg\psi$	Totals
ϕ	$\|m(\phi) \cap m(\psi)\|$	$\|m(\phi) \cap m(\neg\psi)\|$	$\|m(\phi)\|$
$\neg\phi$	$\|m(\neg\phi) \cap m(\psi)\|$	$\|m(\neg\phi) \cap m(\neg\psi)\|$	$\|m(\neg\phi)\|$
Totals	$\|m(\psi)\|$	$\|m(\neg\psi)\|$	$\|U\|$

	ψ	$\neg\psi$	Totals
ϕ	a	b	$a+b$
$\neg\phi$	c	d	$c+d$
Totals	$a+c$	$b+d$	$a+b+c+d=n$

Table 2. Contingency table for rule $\phi \rightarrow \psi$

From the contingency table, we can define some basic quantities. The *generality* of concept ϕ is defined by:

$$g(\phi) = \frac{|m(\phi)|}{|U|} = \frac{a+b}{n}, \tag{35}$$

which indicates the relative size of the concept ϕ. A concept is more general if it covers more instances of the universe. If $g(\phi) = \alpha$, then $(100\alpha)\%$ of objects in U satisfy ϕ. The quantity may be viewed as the probability of a randomly selected element satisfying ϕ. Obviously, we have $0 \leq g(\phi) \leq 1$.

The *absolute support* of ψ provided by ϕ is the quantity:

$$\begin{aligned} as(\psi|\phi) &= \frac{|m(\psi) \cap m(\phi)|}{|m(\phi)|} \\ &= \frac{a}{a+b}. \end{aligned} \tag{36}$$

It may be interpreted as the degree to which ϕ implies ψ. If $as(\psi|\phi) = \alpha$, then $(100\alpha)\%$ of objects satisfying ϕ also satisfy ψ. It is in fact the conditional probability of a randomly selected element satisfying ψ given that the element satisfies ϕ. In set-theoretic terms, it is the degree to which $m(\phi)$ is included in $m(\psi)$. Clearly, $as(\psi|\phi) = 1$, if and only if $m(\phi) \subseteq m(\psi)$. The *change of support* of ψ provided by ϕ is defined by:

$$\begin{aligned} cs(\psi|\phi) &= as(\psi|\phi) - g(\psi) \\ &= \frac{an - (a+b)(a+c)}{(a+b)n}. \end{aligned} \tag{37}$$

Unlike the absolute support, the change of support varies from -1 to 1. One may consider $g(\psi)$ to be the prior probability of ψ and $as(\psi|\phi)$ the posterior probability of ψ after knowing ϕ. The difference of posterior and prior probabilities represents the change of our confidence regarding whether ϕ actually confirms ψ. For a positive value, one may say that ϕ confirms ψ; for a negative value, one may say that ϕ does not confirm ψ. The *mutual support* of ψ and ϕ is defined by:

$$\begin{aligned} ms(\phi,\psi) &= \frac{|m(\phi) \cap m(\psi)|}{|m(\phi) \cup m(\psi)|} \\ &= \frac{a}{a+b+c}. \end{aligned} \tag{38}$$

One may interpret the mutual support, $0 \leq ms(\phi,\psi) \leq 1$, as a measure of the strength of the two-way association $\phi \Leftrightarrow \psi$. It measures the degree to which ϕ confirms, and only confirms, ψ.

The degree of *independence* of ϕ and ψ is measured by:

$$\begin{aligned} ind(\phi,\psi) &= \frac{g(\phi \wedge \psi)}{g(\phi)g(\psi)} \\ &= \frac{an}{(a+b)(a+c)}. \end{aligned} \tag{39}$$

It is the ratio of the joint probability of $\phi \wedge \psi$ and the probability obtained if ϕ and ψ are assumed to be independent. One may rewrite the measure of independence as [3]:

$$ind(\phi,\psi) = \frac{as(\psi|\phi)}{g(\psi)}. \tag{40}$$

It shows the degree of the deviation of the probability of ψ in the subpopulation constrained by ϕ from the probability of ψ in the entire data set [13,30]. With this expression, the relationship to the change of support becomes clear. Instead of using the ratio, the latter is defined by the difference of $as(\psi|\phi)$ and $g(\psi)$. When ϕ and ψ are probabilistic independent, we have $cs(\psi|\phi) = 0$ and $ind(\phi, \psi) = 1$. Moreover, $cs(\psi|\phi) \geq 0$ if and only if $ind(\phi, \psi) \geq 1$, and $cs(\psi|\phi) \leq 0$ if and only if $ind(\phi, \psi) \leq 1$. This provides further support for the use of cs as a measure of confidence that ϕ confirms ψ.

All measures introduced so far have a probabilistic interpretation. They can be roughly divided into three classes: generality (g), one-way association (as and cs), and two-way association (ms and ind). Each type of association measures can be further divided into absolute support and change of support. The measure of absolute one-way support is as, and the measure of absolute two-way support is ms. The measures of change of support are cs for one-way, and ind for two-way. It is interesting to note that all measures of change of support are related to the *deviation* of joint probability of $\phi \wedge \psi$ from the probability obtained if ϕ and ψ are assumed to be independent. In other words, a stronger association is presented if the joint probability is further away from the probability under independence. The association can be either positive or negative.

6 Conclusion

Granular computing may be regarded to as a label of the family of theories, methodologies, and techniques that make use of granules (i.e., groups, classes, or clusters of a universe) in the process of problem solving. The construction, representation, and interpretation of granules, as well as the utilization of granules for problem solving, are some of the fundamental issues. In order to understand and investigate these issues, it is necessary to establish a proper framework. By reviewing some existing studies on non-fuzzy granular computing, we proposed a model of granular computing based on information tables. Within this model, various methods for the construction, interpretation, and representation of granules were examined. Two specific problems of granular computing were also discussed, as an illustration to show the usefulness of our model. One may conclude that although the proposed model is simple, it is powerful for the study of fundamental issues in granular computing.

In this paper, we only considered non-fuzzy granular computing. It is useful to extend the framework so that fuzzy information may be incorporated. This may be done by using fuzzy relations on attribute values. One may have another generalization by considering *incomplete* information tables, where an information function maps each object to a set of attribute values instead of a single value [37]. The major part of our discussion was focused on information granulation, with very little emphasis on the actual computing. Methods for computation based on granulations of universe are clearly needed. The approaches for constructing rough set approximations and finding connections between granules represent two such examples.

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