On Semantic Issues in Game-theoretic Rough Set Model

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Outline

1. Rough Sets
2. Probabilistic Rough Sets
3. Game-theoretic Rough Sets
4. Semantic Issues in GTRS
5. Interpreting an Existing Formulation
Rough Sets

• Sets derived from imperfect, imprecise, and incomplete data may not be able to be precisely defined.

• Sets have to be approximated.

• Approximating a concept $C$ with objects in $U$.
  • Lower approximation given by $\text{apr}(C)\{x \in U | x \subseteq C\}$.
  • Upper approximation given by $\overline{\text{apr}}(C)\{x \in U | x \cap C \neq \emptyset\}$.

• The three regions defined by the approximations.
  • $\text{POS}(C) = \text{apr}(C)$
  • $\text{BND}(C) = \overline{\text{apr}}(C) - \text{apr}(C)$.
  • $\text{NEG}(C) = U - (\text{POS}(C) \cup \text{BND}(C))$. 
Rough Sets

\[ U \]

The Set \( C \)

- Lower approximation
- Upper approximation
Probabilistic Rough Sets

- Defining the approximations in terms of conditional probabilities and a pair of thresholds (Yao, 2008).
  - The \((\alpha, \beta)\) thresholds for determining the probabilistic rough set approximations given by,
    \[
    \text{appr}(\alpha, \beta)(C) = \bigcup \{ [x] \in U/E \mid Pr(C|[x]) \geq \alpha \}, \\
    \text{appr}(\alpha, \beta)(C) = \bigcup \{ [x] \in U/E \mid Pr(C|[x]) > \beta \}. \tag{1}
    \]
  
- Probabilistic positive, negative and boundary regions:
  \[
  \text{POS}(\alpha, \beta)(C) = \{ x \in U \mid Pr(C|[x]) \geq \alpha \}, \\
  \text{NEG}(\alpha, \beta)(C) = \{ x \in U \mid Pr(C|[x]) \leq \beta \}, \\
  \text{BND}(\alpha, \beta)(C) = \{ x \in U \mid \beta < Pr(C|[x]) < \alpha \}. \tag{2}
  \]

Three-way Decisions with Probabilistic Rough Sets

• Three-way decisions are made according to the following rules.

  Acceptance: \( P(C|[x]) \geq \alpha \),

  Rejection: \( P(C|[x]) \leq \beta \), and

  Deferment: \( \beta < P(C|[x]) < \alpha \). \hspace{1cm} (3)

• A major difficulty is the interpretation and determination of the \((\alpha, \beta)\) thresholds (Yao, 2011).

Yao, Y.Y., (2011). Two semantic issues in a probabilistic rough set model. Fundamenta Informaticae 108(3-4).
Determination of \((\alpha, \beta)\) Probabilistic Thresholds

- Realizing the determination of probabilistic thresholds as an optimization based on criterion \(C\).

\[
\arg \max_{(\alpha, \beta)} C(\alpha, \beta), \text{ where }
C(\alpha, \beta) = C_P(\alpha, \beta) + C_N(\alpha, \beta) + C_B(\alpha, \beta).
\]

\[ (4) \]

- Many attempts have been made to determine the thresholds.
  - Optimization viewpoint (Jia et al., 2011),
  - Multi-view model (Li and Zhou, 2011),
  - Method using probabilistic model criteria (Liu et al., 2011),
  - Information-theoretic interpretation (Deng and Yao, 2012),
  - Game-theoretic framework (Herbert and Yao, 2011).

- We consider the game-theoretic rough set model.

Li, H.X., Zhou, X.Z., (2011). Risk decision making based on DTRS... IJCIS 4,
Game Theory

- Game theory is a core subject in decision sciences.
- The basic game components include:
  - Players.
  - Strategies.
  - Payoffs.
- A classical example in Game Theory: The prisoners dilemma.

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<tr>
<td>don’t confess</td>
<td>$p_1$ serves 20 year, $p_2$ serves 0 years</td>
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<tr>
<td>confess</td>
<td>$p_2$ serves 20 years</td>
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<tr>
<td>don’t confess</td>
<td>$p_2$ serves 1 year</td>
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On Semantic Issues in Game-theoretic Rough Set Model

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A Formal Game Definition

• A game may be formally defined as a tuple \( \{P, S, u\} \) (Brown and Shoham, 2008),
  - \( P \) is a finite set of \( n \) players, indexed by \( i \),
  - \( S = S_1 \times ... \times S_n \), where \( S_i \) is a finite set of strategies available to player \( i \). Each vector \( s = (s_1, s_2, ..., s_n) \in S \) is called a strategy profile where player \( i \) selects strategy \( s_i \).
  - \( u = (u_1, ..., u_n) \) where \( u_i : S \rightarrow \mathbb{R} \) is a real-valued utility or payoff function for player \( i \).

Game-theoretic Rough Sets

- Utilizing a game-theoretic setting for analyzing rough sets.
- Determining the probabilistic thresholds to obtain the three regions and the implied three-way decisions.
- Current GTRS based formulations.
  - Game for improving classification ability (Herbert and Yao, 2011).
  - Game for obtaining effective rules (Azam and Yao, 2012).
  - Game for reducing region uncertainties (Azam and Yao, 2013).
  - Game for optimizing Gini Coefficient (Yan, 2011).

Azam, N., Yao, J. T., (2012). Multiple criteria decision analysis with GTRS. In: (RSKT'12).
Azam, N., Yao, J. T., (2013). Analyzing uncertainties of probabilistic rough set regions with GTRS. IJAR.
Yan, Z., (2013). Optimizing GINI coefficient of probabilistic rough set regions using GTRS. In: (CCECE13).
Game-theoretic Rough Sets

\[(s_3 \wedge s_3) = (\alpha_3, \beta_3)\]
Semantic Issues in GTRS

- Interpreting the GTRS based game and its components.
  - Interpreting the players based on application requirements.
  - Understanding strategies based on threshold configuration levels.
  - Strategy profiles and their mappings to probabilistic thresholds.
- Meaning of determined thresholds based on a game outcome.
Interpreting the GTRS based Game

• Implementing a game based on an application needs.
  • The needs may be represented in the form of multiple performance evaluating factors or criteria such as cost, risk, accuracy etc.

• A multi-objective optimization problem may be realized to meet these application needs.

\[
\arg \min_{(\alpha, \beta)} C(\alpha, \beta), \text{ where } \\
C(\alpha, \beta) = (C_1(\alpha, \beta), C_2(\alpha, \beta), ..., C_n(\alpha, \beta)) \quad (5)
\]

• The GTRS based game considers the above optimization as game-theoretic competition or cooperation among multiple criteria.
Interpreting the Players

- Selecting the players to highlight different aspects of application specific needs.

- Example: considering an application which requires an improvement in the classification ability.
  - Accuracy represents one aspect of the requirement.
  - Precision represents another aspect.

- The players may compete or cooperate to reach these game objectives.
Interpreting the Strategies

• Considering strategies as different threshold modification levels.

• Using functions to represent strategies.
  • A strategy $s_i$ of a player $j$ that changes the thresholds, we may use functions to represent $s_i$ as,

  $$s_i = \{(f_i^j(\alpha), g_i^j(\beta)) \mid f_i^j(\alpha) = \alpha \pm c_1, \ g_i^j(\beta) = \beta \pm c_2\}$$  
  
  $c_1, c_2$ are the amount by which we modify the thresholds.

• The threshold values calculated by the functions may be denoted by,

  $$f_i^j(\alpha) = \alpha \pm c_1 = \alpha_i^j$$
  $$g_i^j(\beta) = \beta \pm c_2 = \beta_i^j$$
Mapping a Strategy to a Threshold Pair

- Associating a strategy with a threshold pair.
  - The strategy $s_i$ of player $j$ based on Eq.(6)-(7) can now be associated with $(\alpha_i^j, \beta_i^j)$.

- The functions $(f_i^j, g_i^j)$ provides a mapping that maps each strategy $s_i$ of player $j$ to a threshold pair.

\[
(f_i^j, g_i^j): s_i \longmapsto (D_\alpha, D_\beta),
\]

where $D_\alpha = D_\beta = [0, 1]$ are the domains of thresholds.

- In summary, each strategy leads to a threshold pair.
Interpreting the Strategy Profiles

- Strategy profiles are the possible combination of strategies in a game.
- Considering a special strategy profile \( s = (s_1, s_2, \ldots, s_n) \), where player \( j \) plays \( s_j \).
  - This may be represented in terms of functions defined for individual strategies in equation (6).

\[
s = (s_1, s_2, \ldots, s_n) = ((f_1^1(\alpha), g_1^1(\beta)), \ldots, (f_n^1(\alpha), g_n^1(\beta))) \quad (9)
\]

which leads to threshold pairs,

\[
s = (s_1, s_2, \ldots, s_n) = ((\alpha_1^1, \beta_1^1), \ldots, (\alpha_n^1, \beta_n^1)) \quad (10)
\]
Mapping a Strategy Profile to a Threshold pair

• Realizing a strategy profile and its mapping to a threshold pair.
  • Additional functions may be used for this purpose.

\[ s = (s_1, s_2, ..., s_n) = \{(H_s(\alpha_1^1, \alpha_2^2, ..., \alpha_n^n), O_s(\beta_1^1, \beta_2^2, ..., \beta_n^n))\}, \]

where \( H_s(\alpha_1^1, \alpha_2^2, ..., \alpha_n^n) = \alpha_s, \]
\( O_s(\beta_1^1, \beta_2^2, ..., \beta_n^n) = \beta_s. \) (11)

• The strategy profile \( s \) is now associated with \( (\alpha_s, \beta_s) \).

• The functions \((H_s, O_s)\) maps a strategy profile \( s \) to another threshold pair,

\[ (H_s, O_s): s \mapsto (D_\alpha, D_\beta). \] (12)
The GTRS based Game

- A GTRS based game has now the form \( \{P, S, u\} \), where
  - \( P \) = a finite set of \( n \) players considered as criteria for evaluating application specific requirements.
  - \( S = S_1 \times \ldots \times S_n \), where \( S_j \) is a finite set of strategies available to player \( j \). Each \( s_i \) of player \( j \) maps to a threshold pair by using functions \( f_{ij}^j \) and \( g_{ij}^j \) given by \((f_{ij}^j(\alpha), g_{ij}^j(\beta)) : s_i \mapsto (D_\alpha, D_\beta)\),
  - Each strategy profile of the form \( s = (s_1, s_2, \ldots, s_n) \) also maps to a threshold pair given by \((H_s, O_s) : s \mapsto (D_\alpha, D_\beta)\).
  - \( u = (u_1, \ldots, u_n) \) where \( u_j : S \mapsto \mathbb{R} \) is a real-valued utility or payoff function for player \( j \).
Interpreting the Thresholds Determined by GTRS

• The utility of player $j$ corresponding to the strategy profile $s$ that maps to $(\alpha_s, \beta_s)$ is given by,

$$u_j(s) = u_j(\alpha_s, \beta_s)$$

(13)

• Let $s_{-j} = \{s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n\}$,

  • We may write $s = (s_j, s_{-j})$.
  • The utility of player $j$ becomes

$$u_j(s) = u_j(s_j, s_{-j}) = u_j(\alpha(s_j, s_{-j}), \beta(s_j, s_{-j})).$$
The Game Outcome and the Determined Thresholds

- Interpreting the output or determined thresholds as solution concept of Nash equilibrium.
- Definition of determined thresholds with GTRS.

The GTRS determines a threshold pair that corresponds to a strategy profile $s = (s_1, s_2, ..., s_n) = (s_j, s_{-j})$ such that

$$u_j\left(\alpha(s_j, s_{-j}), \beta(s_j, s_{-j})\right) \geq u_j\left(\alpha(s_j', s_{-j}), \beta(s_j', s_{-j})\right),$$

where $(s_j' \in S_j \land s_j' \neq s_j)$ \hspace{1cm} (14)
Interpreting Herbert and Yao, (2011) Formulation

- The objective was to obtain effective region sizes.
- A competitive game was considered between the probabilistic region parameters $\alpha$ and $\beta$.
- The set of players in the game $P = \{\alpha, \beta\}$.
- Three strategies were considered for player 1, i.e. $\alpha$.
  - The strategy set of player 1 $= S_1 = \{s_1, s_2, s_3\}$, where
    - $s_1 = \text{decrease } \alpha \text{ by 5\%}$,
    - $s_2 = \text{decrease } \alpha \text{ by 7\%}$, and
    - $s_3 = \text{decrease } \alpha \text{ by 15\%}$,
- Similar strategies were defined for player $\beta$.

Interpreting the Strategies

- The strategies may be represented using equation 6.
- Considering the strategies of player $\alpha$.

\[
\begin{align*}
  s_1 &= \{ f^1_1(\alpha) = \alpha - c_1 \times \alpha = \alpha(1 - 0.05) = 0.95\alpha, \ g^1_1(\beta) = \beta \} \\
  s_2 &= \{ f^1_2(\alpha) = \alpha - c_2 \times \alpha = \alpha(1 - 0.07) = 0.93\alpha, \ g^1_2(\beta) = \beta \} \\
  s_3 &= \{ f^1_3(\alpha) = \alpha - c_3 \times \alpha = \alpha(1 - 0.15) = 0.85\alpha, \ g^1_3(\beta) = \beta \}
\end{align*}
\]

- The corresponding threshold pairs are given by,

\[
\begin{align*}
  s_1 &= (\alpha^1_1, \beta^1_1) = (0.95\alpha, \beta) \\
  s_2 &= (\alpha^1_2, \beta^1_2) = (0.93\alpha, \beta), \\
  s_3 &= (\alpha^1_3, \beta^1_3) = (0.85\alpha, \beta).
\end{align*}
\] (16)

- Similar interpretation applies to strategies of $\beta$. 

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Interpreting the Strategy Profiles

- There were nine strategy profiles in this game.

\[ S = S_1 \times S_2 = \{(s_1, s_1), (s_1, s_2), \ldots, (s_3, s_2), (s_3, s_3)\}. \quad (17) \]

- Considering the profile \((s_1, s_1)\), we have

\[
(s_1, s_1) = \{ H_{(s_1, s_1)}(\alpha_1^1, \alpha_1^2) = H_{(s_1, s_1)}(0.95\alpha, \alpha), \\
O_{(s_1, s_1)}(\beta_1^1, \beta_1^2) = O_{(s_1, s_1)}(\beta, 1.05\beta) \} \quad (18)
\]

- The threshold pair corresponding to \((s_1, s_1)\) was determined as,

\[
H_{(s_1, s_1)}(\alpha_1^1, \alpha_1^2) = 0.95\alpha, \quad O_{(s_1, s_1)}(\beta_1^1, \beta_1^2) = 1.05\beta. \quad (19)
\]

- Final threshold values may be determined using the Nash equilibrium solution as defined in Equation 14
Conclusion

- Existing GTRS based formulations and approaches extended the applicability of the model.
- The differences in treatment of game components and determination of thresholds may lead to possible confusions and misinterpretation.
- We address some semantic issues related to the interpretation of game components and the determination of thresholds with GTRS.
- It is hoped that this will improve the understandability of GTRS.
  - Ultimately leading to more interesting applications.
Questions?