A New Inference Axiom for Probabilistic Conditional Independence

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Abstract. In this paper, we present a hypergraph-based inference method for conditional independence. Our method allows us to obtain several interesting results on graph combination. In particular, our hypergraph approach allows us to strengthen one result obtained in a conventional graph-based approach. We also introduce a new inference axiom, called *combination*, of which the contraction axiom is a special case.

1 Introduction

In the design and implementation of a probabilistic reasoning system [5, 6], a crucial issue to consider is the *implication problem* [4]. The implication problem is to test whether a given set of independencies logically implies another independency. Given the set of independencies defining a Bayesian network, the *semi-graphoid inference axioms* [2] can derive every independency holding in the Bayesian network without resorting to their numerical definitions. Shachter [3] has pointed out that this logical system is equivalent to a graphical one involving multiple undirected graphs and some simple graphical transformations. More specifically, every independency used to define the Bayesian network is represented by an undirected graph. The axiomatic derivation of a new independency can then be seen as applying operations on the multiple undirected graphs such as combining two undirected graphs.

In this paper, we present a hypergraph-based inference method for conditional independence. Our method allows us to obtain several interesting results on graph combination, i.e., combining two individual hypergraphs into one single hypergraph. We establish a one-to-one correspondence between the separating sets in the combined hypergraph and certain separating sets in one of the individual hypergraphs. In particular, our hypergraph approach allows us to strengthen one result obtained by Shachter in the graph-based approach. Moreover, our analysis leads us to introduce a new inference axiom, called *combination*, of which the contraction axiom is a special case.

This paper is organized as follows. In Section 2, we review two pertinent notions. In Section 3, we introduce the notion of hypergraph combination. The combination inference axiom is introduced in Section 4. In Section 5, we present our main result. The conclusion is presented in Section 6.

2 Background Knowledge

In this section, we review the pertinent notions of probabilistic conditional independence and hypergraphs used in this study.

Let $R = \{A_1, A_2, \ldots, A_m\}$ denote a finite set of discrete variables. Each variable A_i is associated with a finite domain D_i. Let D be the Cartesian product of the domains $D_1, \ldots, D_m$. A *joint probability distribution* on D is function p on D, $p : D \rightarrow [0, 1]$, such that p is normalized. That is, this function p assigns to each tuple $t \in D$ a real number $0 \leq p(t) \leq 1$ and $\sum_{t \in D} p(t) = 1$. We write a joint probability distribution p as $p(A_1, A_2, \ldots, A_m)$ over the set R of variables.

Let X, Y, and Z be disjoint subsets of R. Let x, y and z be arbitrary values of X, Y and Z, respectively. We say Y and Z are *conditionally independent* given X under the joint probability distribution p, denoted $I(Y, X, Z)$, if $p(y|x, z) = p(y|x)$, whenever $p(x, z) > 0$. We call an independency $I(Y, X, Z)$ *full*, in the special case when $XYZ = R$. In the probabilistic reasoning theory, probabilistic conditional independencies are often graphically represented using hypergraphs.

A *hypergraph* [1] $\mathcal{H}$ on a finite set R of vertices is a set of subsets of R, that is, $\mathcal{H} = \{R_1, R_2, \ldots, R_n\}$, where $R_i \subseteq R$ for $i = 1, 2, \ldots, n$. (Henceforth, we will simply refer to the hypergraph $\mathcal{H}$ and assume $R = R_1 \cup R_2 \cup \ldots \cup R_n$). For example, two hypergraphs $\mathcal{H}_1 = \{h_1 = \{A, B, C, D, E\},\ h_2 = \{D, E, F\}\}$ and $\mathcal{H}_2 = \{\{A, B\}, \{A, C\}, \{B, D\}, \{C, E\}\}$ are shown in Figure 1 (i).

A hypergraph $\mathcal{H} = \{R_1, R_2, \ldots, R_n\}$ is *acyclic* [1], if there exists a permutation $S_1, S_2, \ldots, S_n$ of $R_1, R_2, \ldots, R_n$ such that for $j = 2, ..., n$, $S_j \cap (S_1 \cup S_2 \cup \ldots \cup S_{j-1}) \subseteq S_i$, where $i < j$. It can be verified that the two hypergraphs in Figure 1 (i) are each acyclic, whereas the one in Figure 1 (ii) is not.

If $\mathcal{H}$ is a hypergraph, then the set of conditional independencies *generated by* $\mathcal{H}$ is the set $CI(\mathcal{H})$ of full conditional independencies $I(Y, X, Z)$, where Y is the union of some disconnected components of the hypergraph $\mathcal{H} - X$ obtained from $\mathcal{H}$ by deleting the set X of nodes, and $Z = R - XY$. That is, $\mathcal{H} - X = \{h - X \mid h \text{ is a hyperedge of } \mathcal{H}\} - \{\emptyset\}$. We then say that X *separates off* Y from the rest of the nodes, and call X a *separating set* [1]. For example, consider the acyclic hypergraph $\mathcal{H}_2$ in Figure 1 (i). If $X = \{B, C\}$, then $\mathcal{H} - X = \{\{A\}, \{D\}, \{E\}\}$. By definition, $I(A, BC, DE)$ and $I(D, BC, AE)$ are two conditional independencies appearing in $CI(\mathcal{H})$.

The conditional independencies generated by a hypergraph $\mathcal{H}$, i.e., $CI(\mathcal{H})$, can be equivalently expressed using conventional undirected graphs and the separation method [1]. If H is a hypergraph, then the *graph of* H, denoted $G(H)$, is defined as: $G(H) = \{\ (A, B) \mid A \in h \text{ and } B \in h \text{ for some } h \in H\ \}$. For instance, the graph of the hypergraph $\mathcal{H}$ in Figure 1 (ii) is the undirected graph $G(\mathcal{H}) = \{(A, B), (A, C), (B, D), (C, E), (D, E), (D, F), (E, F)\}$. It can be verified that $CI(\mathcal{H}) = CI(G(\mathcal{H}))$.

3 Hypergraph Combination

This section focuses on the graphical combination of two individual hypergraphs into a single hypergraph.

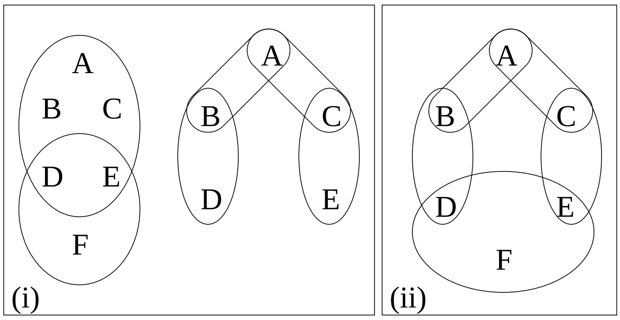

Fig. 1. Given $\mathcal{H}_1 = \{h_1 = \{A, B, C, D, E\}, h_2 = \{D, E, F\}\}$ and $\mathcal{H}_2 = \{\{A, B\}, \{A, C\}, \{B, D\}, \{C, E\}\}$ in (i), the *combination* of $\mathcal{H}_1$ and $\mathcal{H}_2$ is $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ in (ii).

Let $X_1Y_1Z_1 = R$ such that X_1, Y_1 and Z_1 are pairwise disjoint and each nonempty. Let $\mathcal{H}_1 = \{h_1 = Y_1X_1,\ h_2 = X_1Z_1\}$ be a binary acyclic hypergraph and $\mathcal{H}_2$ be any hypergraph defined on the set h_1 of variables. The *combination* of $\mathcal{H}_1$ and $\mathcal{H}_2$, written $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$, is defined as:

$$\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2} = (\mathcal{H}_1 - \{h_1\})\ \cup\ \mathcal{H}_2, \tag{1}$$

or equivalently,

$$\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2} = \mathcal{H}_2 \cup \{h_2\}. \tag{2}$$

Example 1. Consider the two acyclic hypergraphs $\mathcal{H}_1 = \{h_1 = \{A, B, C, D, E\}, h_2 = \{D, E, F\}\}$ and $\mathcal{H}_2 = \{\{A, B\}, \{A, C\}, \{B, D\}, \{C, E\}\}$ in Figure 1 (i). The combination of $\mathcal{H}_1$ and $\mathcal{H}_2$ is the hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2} = \{\{A, B\}, \{A, C\}, \{B, D\}, \{C, E\}, \{D, E, F\}\}$, as depicted in Figure 1 (ii).

As Shachter pointed out in [3], a set X may be a separating set in the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ but not in $\mathcal{H}_1$. The set BE separates AC and DF in the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ of Figure 1 (ii), but not in hypergraph $\mathcal{H}_1$ of Figure 1 (i). The next result precisely characterize the new separating sets.

Lemma 1. Let $\mathcal{H}_1 = \{Y_1X_1, X_1Z_1\}$ and $\mathcal{H}_2 = \{Y_2X_2, X_2Z_2\}$, where $X_1Y_1 = X_2Y_2Z_2$. If $Y_2 \cap X_1 = \emptyset$, then X_2 separates Y_2 and Z_1Z_2 in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$.

Proof: Suppose $Y_2 \cap X_1 = \emptyset$. Since $X_1Y_1 = X_2Y_2Z_2$, we have $X_1 \subseteq X_2Z_2$. Thus, X_1 can be augmented by some subset $W \subseteq Y_1$ to be equal to X_2Z_2, namely, $X_1W = X_2Z_2$. Thus, X_1 separating Y_1 and Z_1 in $\mathcal{H}_1$ can be restated as X_1 separates Y_2W and Z_1 in $\mathcal{H}_1$. It immediately follows that X_1W separates Y_2 and Z_1 in $\mathcal{H}_1$. Since $X_1W = X_2Z_2$, we have X_2Z_2 separates Y_2 and Z_1 in $\mathcal{H}_1$. By *graphical contraction* [3], X_2 separating Y_2 and Z_2 in $\mathcal{H}_2$ and X_2Z_2 separating Y_2 and Z_1 in $\mathcal{H}_1$, implies that X_2 separates Y_2 and Z_1Z_2 in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$. □

Lemma 1 can be understood using the database notion of *splits* [1]. Given a hypergraph $\mathcal{H}$, a set X *splits* two variables A and B, if X blocks every path between A and B. More generally, a set X splits a set W, if X splits at least two attributes of W. Lemma 1 then means that if the separating set X_2 does *not* split X_1, then X_2 will remain a separating set in the combined hypergraph.

Example 2. Consider again Figure 1. Here DE is the only separating set of $\mathcal{H}_1$. The separating set B of $\mathcal{H}_2$ in (i) *splits* DE, since D is separated from E by B. By Lemma 1, B will not be a separating set in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$. On the other hand, the separating set BE of $\mathcal{H}_2$ in (ii) will indeed be a separating set in the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ since BE does *not* split DE.

Lemma 2. There is a one-to-one correspondence between the separating sets of $\mathcal{H}_2$ that do *not* split X_1 and the *new* separating sets in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$.

Proof: Suppose X_2 separates Y_2 and $Z_1 Z_2$ in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$. It follows that X_2 separates Y_2 and Z_2 in the hypergraph obtained by projecting down to the context $X_2 Y_2 Z_2$, namely, $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2} - Z_1 = \{h - Z_1 \mid h \in \mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}\} = \mathcal{H}_2 \cup \{X_1\}$. It is well-known [1, 2] that removing a hyperedge from a hypergraph can only add new separating sets, i.e., removing a hyperedge from a hypergraph cannot destroy an existing separating set. Thus, since X_2 separates Y_2 and Z_2 in $\mathcal{H}_2 \cup \{X_1\}$, X_2 separates Y_2 and Z_2 in the smaller hypergraph $\mathcal{H}_2$. $\Box$

Example 3. All of the separating sets of $\mathcal{H}_2$ are listed in the first column of Table 1. The horizontal line partitions those separating sets that do not split DE from those that do. Those separating sets that do *not* split DE are listed above the horizontal line, while those that split DE are listed below. The *new* separating sets in the combined hypergraph are given in the 3rd column. (The fact that DE separates F and ABC is already known from $\mathcal{H}_1$.) As indicated, there is a one-to-one correspondence between the separating sets in the smaller hypergraph $\mathcal{H}_2$ that do *not* split DE and the previously unknown separating sets in the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$.

4 The Combination Inference Axiom

Pearl's *semi-graphoid axiomatization* [2] is:

$$
\begin{aligned}
&(SG1)\ Symmetry: && I(Y,X,Z) \Longrightarrow I(Z,X,Y),\\
&(SG2)\ Decomposition: && I(Y,X,ZW) \Longrightarrow I(Y,X,Z)\ \&\ I(Y,X,W),\\
&(SG3)\ Weak\ union: && I(Y,X,ZW) \Longrightarrow I(Y,XZ,W),\\
&(SG4)\ Contraction: && I(W,XY,Z)\ \&\ I(Y,X,Z) \Longrightarrow I(WY,X,Z).
\end{aligned}
$$

We introduce *combination* (SG5) as a new inference axiom for CI:

$$(SG5)\ Combination: \ I(Y_2,X_2,Z_2)\ \&\ I(Y_1,X_1,Z_1) \Longrightarrow I(Y_2,X_2,Z_1Z_2),$$

where $X_1Y_1 = X_2Y_2Z_2$ and $I(Y_2,X_2,Z_2)$ does not split X_1.

Table 1. There is a one-to-one correspondence between the separating sets in $\mathcal{H}_2$ that do *not* split DE and the *new* separating sets in the combined hypergraph.

	separating sets of $\mathcal{H}_2$		separating sets in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$
	$I(B, AD, CE)$	$\longleftrightarrow$	$I(B, AD, CEF)$
DE	$I(C, AE, BD)$	$\longleftrightarrow$	$I(C, AE, BDF)$
is	$I(A, BC, DE)$	$\longleftrightarrow$	$I(A, BC, DEF)$
not	$I(AC, BE, D)$	$\longleftrightarrow$	$I(AC, BE, DF)$
split	$I(AB, CD, E)$	$\longleftrightarrow$	$I(AB, CD, EF)$
	$I(A, BCD, E)$	$\longleftrightarrow$	$I(A, BCD, EF)$
	$I(A, BCE, D)$	$\longleftrightarrow$	$I(A, BCE, DF)$
	$I(BD, A, CE)$		-
DE	$I(D, B, ACE)$		-
is	$I(E, C, ABD)$		-
split	$I(D, AB, CE)$		-
	$I(BD, AC, E)$		-
	$I(D, BC, AE)$		-

Lemma 3. The *combination* inference axiom (SG5) is *sound* for probabilistic conditional independence.

Proof: Since $I(Y_2, X_2, Z_2)$ does not split X_1, at least one of Y_2 or Z_2 does not intersect with X_1. Without loss of generality, let $Y_2 \cap X_1 = \emptyset$.

By the proof of Lemma 1, $I(Y_1, X_1, Z_1)$ can be rewritten as $I(Y_2W, X_1, Z_1)$. By (SG3), we obtain $I(Y_2, X_1W, Z_1)$. Since $X_1W = X_2Z_2$, we have $I(Y_2, X_2Z_2, Z_1)$. By (SG4), $I(Y_2, X_2, Z_2)$ and $I(Y_2, X_2Z_2, Z_1)$ give $I(Y_2, X_2, Z_1Z_2)$. $\Box$

Lemma 3 indicates that $\{(SG1), (SG2), (SG3), (SG4)\} \Longrightarrow (SG5)$. The next result shows that (SG4) and (SG5) can be interchanged.

Theorem 4. $\{(SG1), (SG2), (SG3), (SG5)\} \Longrightarrow (SG4)$.

Proof: We need to show that any CI obtained by (SG4) can be obtained using $\{(SG1), (SG2), (SG3), (SG5)\}$. Suppose we are given $I(W, XY, Z)$ and $I(Y, X, Z)$. By (SG5), we obtain the desired CI $I(WY, X, Z)$. $\Box$

Corollary 5. The contraction inference axiom is a *special case* of the combination inference axiom.

By Theorem 4 and Lemma 3, we have:

$$\{(SG1), (SG2), (SG3), (SG4)\} \equiv \{(SG1), (SG2), (SG3), (SG5)\}.$$

The combination axiom can be used for *convenience.* For instance, consider deriving $I(AC, BE, DF)$ from $I(F, DE, ABC)$ and $I(D, BE, AC)$. Using the semi-graphoid axiomatization $\{(SG1), (SG2), (SG3), (SG4)\}$ requires four steps, whereas using $\{(SG1), (SG2), (SG3), (SG5)\}$ requires three steps.

5 Reasoning with Multiple Hypergraphs

In this section, we focus on those sets $\sum$ of CIs, where the semi-graphoid axioms are complete. That is, $\sum$ logically implies another independency σ if and only if σ can be derived from $\sum$ by applying the four inference axioms $\{(SG1), (SG2), (SG3), (SG4)\}$.

The main result is that the combination $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ is a *perfect-map* of the full conditional independencies logically implied by the independencies in $\mathcal{H}_1$ together with those in $\mathcal{H}_2$.

A hypergraph $\mathcal{H}$ is an *independency-map* (I-map) [2] for a joint distribution $p(R)$, if every independency $I(Y, X, Z)$ in $CI(\mathcal{H})$ is satisfied by $p(R)$. A hypergraph $\mathcal{H}$ is a *perfect-map* (P-map) [2] for a joint distribution $p(R)$, if an independency $I(Y, X, Z)$ is in $CI(\mathcal{H})$ if and only if it is satisfied by $p(R)$.

In [3], Shachter established the following:

$$I(Y, X, Z) \in CI(\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}) \implies CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2) \models I(Y, X, Z),$$

that is, if X separates Y and Z in the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$, then $I(Y, X, Z)$ is logically implied by $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2)$. Theorem 6 below shows that a CI $I(Y, X, Z)$ can be inferred by separation in the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ iff $I(Y, X, Z)$ is logically implied by $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2)$, namely,

$$I(Y, X, Z) \in CI(\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}) \iff CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2) \models I(Y, X, Z).$$

Theorem 6. The combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ is a *perfect-map* of the full conditional independencies logically implied by $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2)$.

Proof: ($\Rightarrow$) Let $I(Y, X, Z) \in CI(\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2})$. Suppose $I(Y, X, Z) \in CI(\mathcal{H}_1)$. Then $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2) \models I(Y, X, Z)$. Suppose then that $I(Y, X, Z) \notin CI(\mathcal{H}_1)$. Since X does not separate Y and Z in $\mathcal{H}_1$, X must necessarily separate two nonempty sets Y_2 and Z_2 in $\mathcal{H}_2$. That is, $I(Y_2, X, Z_2) \in CI(\mathcal{H}_2)$. There are two cases to consider. Suppose $I(Y_2, X, Z_2)$ does *not* split X_1. Without loss of generality, let $Y_2 \cap X_1 = \emptyset$. By Lemma 3, $I(Y_2, X, Z_2)$ and $I(Y_1, X_1, Z_1)$ logically imply $I(Y_2, X, Z_1Z_2)$. Thus, $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2) \models I(Y_2, X, Z_1Z_2)$. We now show that $I(Y_2, X, Z_1Z_2) = I(Y, X, Z)$. Since $I(Y, X, Z) \in CI(\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2})$, deleting X in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ gives two disconnected components Y and Z. Similarly, $I(Y_2, X, Z_1Z_2) \in CI(\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2})$ means that deleting X in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ gives two disconnected components Y_2 and Z_1Z_2. By definition, however, the disconnected components in $\mathcal{H} - W$ are *unique* for any hypergraph on R and $W \subseteq R$. Thus, either $Y = Y_2$ and $Z = Z_1Z_2$ or $Y = Z_1Z_2$ and $Z = Y_2$. In either case, $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2) \models I(Y_2, X, Z_1Z_2)$. Now suppose $I(Y_2, X, Z_2)$ *splits* X_1. By the definition of splits, $X_1 \cap Y_2 \neq \emptyset$ and $X_1 \cap Z_2 \neq \emptyset$. But then contraction can never be applied:

$$I(Y_2, X, Z_2) \,\&\, I(Y_1 - W, X_1W, Z_1) \not\models I(Y_2, X, Z_1Z_2),$$

since $X_1W \neq XZ_2$ for every subset $W \subseteq Y_1$.

(⇐) Suppose $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2) \models I(Y, X, Z)$. If $I(Y, X, Z) \in CI(\mathcal{H}_1)$, then X separates Y and Z in $\mathcal{H}_1$ and subsequently in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$. Suppose then that $I(Y, X, Z) \notin CI(\mathcal{H}_1)$. Since $\{(SG1), (SG2), (SG3), (SG4)\}$ can derive every logically implied independency and $\{(SG1), (SG2), (SG3)\}$ are all defined with respect to the same fixed set of variables, the contraction axiom $(SG4)$ must have been applied to derive $I(Y, X, Z)$, i.e,

$$I(Y_1, X_1, Z_1) \ \& \ I(Y_2, X_2, Z_2) \ \models I(Y, X, Z).$$

By definition of $(SG4)$, $I(Y_2, X_2, Z_2)$ does not split X_1, $X_2Y_2Z_2 = X_1Z_1$, $X = X_2$, $Y = Y_1Y_2$, and $Z = Z_1 = Z_2$. Interpreting the conditional independence statement $I(Y_1Y_2, X_2, Z_1)$ as a separation statement means that $X = X_2$ is a separator in the combined graph. That is, X separates Y and Z in $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$. Therefore, $I(Y, X, Z) \in CI(\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2})$. □

6 Conclusion

This study emphasizes the usefulness of viewing graph combination from a hypergraph perspective rather than from a conventional undirected graph approach. Whereas it was previously shown by Shachter [3] that the combined hypergraph $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ is an I-map of the full conditional independencies logically implied by $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2)$, Theorem 6 shows that $\mathcal{H}_1^{h_1 \leftarrow \mathcal{H}_2}$ is in fact a P-map of the full conditional independencies logically implied by $CI(\mathcal{H}_1) \cup CI(\mathcal{H}_2)$. Moreover, in Lemma 2, we were able to draw a one-to-one correspondence between the new separating sets in the combined hypergraph with the separating sets in the smaller hypergraph. Finally, our study of graphical combination lead to the introduction of a new inference axiom for conditional independence, called *combination*, which is a generalization of contraction as Corollary 5 establishes.

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