A Comparative Study of Noncontextual and Contextual Dependencies

S.K.M. Wong[1] and C.J. Butz[2]

[1] Department of Computer Science, University of Regina
Regina, Saskatchewan, Canada S4S 0A2
E-mail: wong@cs.uregina.ca

[2] School of Information Technology & Engineering, University of Ottawa
Ottawa, Ontario, Canada K1N 6N5
E-mail: butz@site.uottawa.ca

Abstract. There is current interest in generalizing Bayesian networks by using dependencies which are more general than *probabilistic conditional independence* (CI). *Contextual* dependencies, such as *context-specific independence* (CSI), are used to decompose a subset of the joint distribution. We have introduced a more general contextual dependency than CSI, as well as a more general noncontextual dependency than CI. We developed these probabilistic dependencies based upon a new method of expressing database dependencies. By defining database dependencies using *equivalence relations*, the difference between the various contextual and noncontextual dependencies can be easily understood. Moreover, this new representation of dependencies provides a convenient tool to readily derive other results.

1 Introduction

Bayesian networks [5] have become an established framework for uncertainty management in artificial intelligence. Bayesian networks only use a single type of dependency, called *probabilistic conditional independence* (CI), to losslessly decompose a joint probability distribution. There is current interest, however, in generalizing Bayesian networks with more general dependencies. In [1], a *contextual* (horizontal) dependency, called *context-specific independence* (CSI), was introduced to capture CIs that only hold in some of the tuples in a joint distribution. In [8], we introduced a more general contextual dependency than CSI, as well as a more general noncontextual dependency than CI. The important point, however, is that our probabilistic dependencies were motivated by corresponding database dependencies.

Weak multivalued dependency (WMVD) [2,3] is a more general database dependency than *multivalued dependency* (MVD) [4]. Fischer and Van Gucht [2] gave several characterizations of WMVD. In this paper, we suggest a new characterization of both MVD and WMVD based on *equivalence relations*. In this framework, the difference between the various database dependencies can be easily understood. Moreover, this new representation of dependencies provides a convenient tool to readily derive other results.

This paper is organized as follows. In Section 2, we review some pertinent notions in the relational database model, and recall some notions about equivalence relations. We use this framework to express contextual and noncontextual dependencies in Section 3. In Section 4, we demonstrate the simplicity of our framework by showing the soundness of some known inference axioms. The conclusion is given in Section 5.

2 Basic Notions

2.1 Relational Databases

Here we review some notions used in the elegant relational database model [4].

A *relation scheme* $R = \{A_1, A_2, \ldots, A_m\}$ is a finite set of attributes. Corresponding to each attribute A_i is a nonempty finite set D_i, $1 \leq i \leq m$, called the *domain* of A_i. Let $D = D_1 \cup D_2 \ldots \cup D_m$. A *relation* r on the relation scheme R, written $r(R)$, is a finite set of mappings $\{t_1, t_2, \ldots, t_s\}$ from R to D with the restriction that for each mapping $t \in r$, $t(A_i)$ must be in D_i, $1 \leq i \leq k$, where $t(A_i)$ denotes the value obtained by restricting the mapping t to A_i. The mappings are called *tuples* and $t(A)$ is called the A-value of t. We use $t(X)$ in the obvious way and call it the X-value of t.

Mappings are used in our exposition to avoid any explicit ordering of the attributes in the relation scheme. To simplify the notation, however, we will henceforth denote relations by writing the attributes in a certain order and the tuples as lists of values in the same order. Furthermore, the following relational database conventions will be adopted for simplified notation. Uppercase letters A, B, C from the beginning of the alphabet may be used to denote attributes. A relation scheme $R = \{A_1, A_2, \ldots, A_m\}$ may be written as simply $A_1A_2 \ldots A_m$. A relation r on scheme R is then written as either $r(R)$ or $r(A_1A_2 \ldots A_m)$. The singleton set $\{A\}$ is sometimes written as A and concatenation XY may be used to denote set union $X \cup Y$.

The *select* σ, *project* π, and *natural join* $\bowtie$ operators are defined as follows.

When the *select* operator σ is applied to a relation r, it yields another relation that is a subset of tuples of r with a certain value on a specified attribute. Let r be a relation on scheme R, $A \in R$, and $a \in D_A$. Then

$$\sigma_{A=a}(r) = \{t \mid t \in r \text{ and } t(A) = a\}.$$

Whereas the select operator chooses a subset of tuples in a relation, the *project* operator π chooses a subset of attributes. Let r be a relation on R and X a subset of R. The *projection of* r *onto* X, written $\pi_X(r)$, is defined as

$$\pi_X(r) = \{\ t(X) \mid t \in r\ \}. \tag{1}$$

The *natural join* of two relations $r_1(X)$ and $r_2(Y)$, written $r_1(X) \bowtie r_2(Y)$, is defined as

$$r_1(X) \bowtie r_2(Y)i = \{\ t(XY) \mid t(X) \in r_1(X) \text{ and } t(Y) \in r_2(Y)\ \}. \tag{2}$$

A fundamental database dependency, namely, *multivalued dependency* (MVD), can now be defined.

Definition 1. Let X, Y, Z be pairwise disjoint subsets of scheme $R = XYZ$. A relation $r(XYZ)$ satisfies the *multivalued dependency* MVD(Y,X,Z), if for any two tuples t_1 and t_2 in r with $t_1(X) = t_2(X)$, there exists a tuple t_3 in r with $t_3(XY) = t_1(XY)$ and $t_3(Z) = t_2(Z)$.

The multivalued dependency MVD(Y,X,Z) is a *necessary* and *sufficient* condition for $r(XYZ)$ to be losslessly decomposed as

$$r(XYZ) = \pi_{XY}(r) \bowtie \pi_{XZ}(r). \tag{3}$$

Example 2. The following relation $r_1(ABC)$ on the left satisfies the multivalued dependency MVD(A,B,C), since

$$r_1(ABC) = \pi_{AB}(r_1) \bowtie \pi_{BC}(r_1).$$

However, $r_2(ABC)$ on the right does *not* satisfy MVD(A,B,C) since

$$r_2(ABC) \neq \pi_{AB}(r_2) \bowtie \pi_{BC}(r_2).$$

$$r_1(ABC) = \begin{array}{|ccc|}\hline A & B & C \\ \hline 0&0&0\\ 0&0&1\\ 1&0&0\\ 1&0&1\\ 1&1&0\\ 1&1&1\\ \hline\end{array} = \begin{array}{|cc|}\hline A&B\\ \hline 0&0\\ 1&0\\ 1&1\\ \hline\end{array} \bowtie \begin{array}{|cc|}\hline B&C\\ \hline 0&0\\ 0&1\\ 1&0\\ 1&1\\ \hline\end{array}, \quad r_2(ABC) = \begin{array}{|ccc|}\hline A&B&C\\ \hline 0&0&0\\ 1&0&1\\ 1&1&1\\ \hline\end{array} \neq \begin{array}{|cc|}\hline A&B\\ \hline 0&0\\ 1&0\\ 1&1\\ \hline\end{array} \bowtie \begin{array}{|cc|}\hline B&C\\ \hline 0&0\\ 0&1\\ 1&1\\ \hline\end{array} = \begin{array}{|ccc|}\hline A&B&C\\ \hline 0&0&0\\ 0&0&1\\ 1&0&0\\ 1&0&1\\ 1&1&1\\ \hline\end{array}$$

2.2 Properties of Equivalence Relations

Since we suggest that dependencies can be conveniently expressed using equivalence relations, we first recall some familiar notions about relations [6].

Given any subset $X \subseteq R$, we can define an *equivalence relation* $\theta(X)$ on r (a partition of r): for all $t_i, t_j \in r$,

$$t_i \ \theta(X) \ t_j, \quad \text{if } t_i(X) = t_j(X). \tag{4}$$

The *composition* operator $\circ$ is used to combine relations. Let $T = \{t_1, t_2, \ldots, t_s\}$ denote a finite set of objects. Consider two relations θ_1 and θ_2 on T. The binary operator $\circ$, called the *composition*, is defined by: for $t_i, t_k \in T$,

$$t_i(\theta_1 \circ \theta_2)t_k, \quad \text{if for some } t_j \in T \text{ both } t_i\theta_1 t_j \text{ and } t_j\theta_2 t_k. \tag{5}$$

It can be shown that the composition $\theta_1 \circ \theta_2$, of two individual equivalence relations θ_1 and θ_2, is itself an equivalence relation (a partition) if and only if $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

We can then define MVD using equivalence relations as follows:

Definition 3. Relation $r(XYZ)$ satisfies MVD(Y,X,Z), if

$$\theta(X) \quad = \quad \theta(XY) \circ \theta(XZ) \quad = \quad \theta(XZ) \circ \theta(XY). \tag{6}$$

3 Generalizing Multivalued Dependency

In this section, we generalize MVD with both *contextual* and *noncontextual* dependencies. Contextual dependencies only decompose a subset of the relation, while noncontextual dependencies decompose the entire relation.

3.1 Context Strong Multivalued Dependency (CSMVD)

Sometimes only a few tuples in a relation cause the violation of an MVD. In this section, we introduce *context strong multivalued dependency* (CSMVD) in order to losslessly decompose *part* (a subset) of a relation.

Consider the relation $r_1(ABC)$ in Figure 1. It can be verified that $r_1(ABC)$ does *not* satisfy MVD(A,B,C). The reason is because the definition of MVD requires that MVD(Y,X=x,Z) holds for *all* X-values x in relation $r(XYZ)$. In this example, this means that MVD(A,B=0,C) and MVD(A,B=1,C) must both hold. However, it can be seen that the MVD(A,B,C) holds when B=0, but *not* when B=1. The important point is that even though the entire relation $r_1(ABC)$ cannot be losslessly decomposed using MVD, namely,

$$r_1(ABC) \quad \neq \quad \pi_{AB}(r_1) \bowtie \pi_{BC}(r_1),$$

it is still possible to losslessly decompose the tuples $\sigma_{B=0}(r_1) = \{t_1, t_2, t_3, t_4\}$:

$$\sigma_{B=0}(r_1) \quad = \quad \pi_{AB}(\sigma_{B=0}(r_1)) \bowtie \pi_{BC}(\sigma_{B=0}(r_1)).$$

$r_1(ABC) =$

A	B	C
0	0	0
0	0	1
1	0	0
1	0	1
0	1	0
0	1	1
1	1	1

, $r_2(ABC) =$

A	B	C
0	0	0
0	0	1
1	0	0
1	0	1
2	0	2
3	0	2
0	1	0
1	1	0

, $r_3(ABC) =$

A	B	C
0	0	0
0	0	1
1	0	0
1	0	1
2	0	2
2	0	3
3	0	2
1	1	1

Fig. 1. Relation $r_1(ABC)$ satisfies CSMVD(A,B=0,C). Relation $r_2(ABC)$ satisfies WMVD(A,B,C). Relation $r_3(ABC)$ satisfies CWMVD(A,B=0,C).

Definition 4. Relation $r(XYZ)$ satisfies the *context strong multivalued dependency* CSMVD(Y,X=x,Z), if the equivalence class defined by $X = x$ in the equivalence relation $\theta(X)$ satisfies the following condition:

$$\theta(X = x) \quad = \quad \theta(X = xY) \circ \theta(X = xZ) \quad = \quad \theta(X = xZ) \circ \theta(X = xY). \quad (7)$$

Example 5. Context Strong Multivalued Dependency. Let us verify that relation $r_1(ABC)$ in Figure 1 satisfies CSMVD(A,B=0,C). By Equation (4), we first obtain

$$\theta(B = 0) \quad = \quad \{[t_1, t_2, t_3, t_4]\}.$$

By another application of Equation (4), we obtain the equivalence relations:

$$\theta(AB = 0) \quad = \quad \{[t_1, t_2], [t_3, t_4]\},$$

and

$$\theta(B = 0C) \quad = \quad \{[t_1, t_3], [t_2, t_4]\}.$$

Applying Equation (5) gives:

$$\theta(AB = 0) \circ \theta(B = 0C) \quad = \quad \{[t_1, t_2, t_3, t_4]\} \quad = \quad \theta(B = 0C) \circ \theta(AB = 0).$$

We have our desired result since:

$$\theta(B = 0) \quad = \quad \theta(AB = 0) \circ \theta(B = 0C) \quad = \quad \theta(B = 0C) \circ \theta(AB = 0).$$

However, it can be similarly verified that CSMVD(A,B=1,C) is *not* satisfied.

CSMVD generalizes MVD by only decomposing *some* of the tuples in a relation. However, it is also possible to generalize MVD with a noncontextual dependency, called *weak multivalued dependency* (WMVD), which decomposes *all* of the tuples in a relation.

3.2 Weak Multivalued Dependency (WMVD)

Weak multivalued dependency (WMVD) [3] generalizes MVD(Y,X,Z) in Definition (3) by not requiring the equivalence relation $\theta(XY) \circ \theta(XZ) \ = \ \theta(XZ) \circ \theta(XY)$ to be equal to $\theta(X)$.

Definition 6. Relation $r(XYZ)$ satisfies the *weak multivalued dependency* WMVD(Y,X,Z), if

$$\theta(XY) \circ \theta(XZ) = \theta(XZ) \circ \theta(XY). \quad (8)$$

Example 7. Weak Multivalued Dependency. Let us verify that relation $r_2(ABC)$ in Figure 1 satisfies WMVD(A,B,C). By Equation (4), we obtain:

$$\theta(AB) = \{[t_1, t_2], [t_3, t_4], [t_5], [t_6], [t_7], [t_8]\},$$

and

$$\theta(BC) = \{[t_1, t_3], [t_2, t_4], [t_5, t_6], [t_7, t_8]\}.$$

Applying Equation (5), we obtain our desired result since:

$$\theta(AB) \circ \theta(BC) = \{[t_1, t_2, t_3, t_4], [t_5, t_6], [t_7, t_8]\} \ = \ \theta(BC) \circ \theta(AB).$$

Thus, even though a relation does not satisfy MVD(Y,X,Z), it may still be possible to losslessly decompose the *entire* relation using WMVD(Y,X,Z).

3.3 Context Weak Multivalued Dependency (CWMVD)

We can introduce a contextual version of WMVD, called *context weak multivalued dependency* (CWMVD).

Definition 8. Relation $r(XYZ)$ satisfies the *context weak multivalued dependency* CWMVD(Y,X=x,Z), if there exists a maximal disjoint compatibility class $\{t_i, \ldots, t_j\}$ in the relation $\theta(X = xY) \circ \theta(X = xZ)$.

Definition 8 implies that $\{t_i, \ldots, t_j\}$ satisfies MVD(Y,X,Z).

Example 9. Context Weak Multivalued Dependency. To verify that relation $r_3(ABC)$ in Figure 1 satisfies WMVD(A,B=0,C), we first obtain:

$$\theta(AB = 0) = \{[t_1, t_2], [t_3, t_4], [t_5, t_6], [t_7]\},$$

and

$$\theta(B = 0C) = \{[t_1, t_3], [t_2, t_4], [t_5, t_7], [t_6]\}.$$

Applying Equation (5), we obtain $\mathcal{R} = \theta(AB = 0) \circ \theta(B = 0C)$:

$$\begin{aligned} \mathcal{R} \quad = \quad & \{t_1\mathcal{R}t_1, t_1\mathcal{R}t_2, t_1\mathcal{R}t_3, t_1\mathcal{R}t_4, t_2\mathcal{R}t_1, t_2\mathcal{R}t_2, t_2\mathcal{R}t_3, t_2\mathcal{R}t_4, \\ & t_3\mathcal{R}t_1, t_3\mathcal{R}t_2, t_3\mathcal{R}t_3, t_3\mathcal{R}t_4, t_4\mathcal{R}t_1, t_4\mathcal{R}t_2, t_4\mathcal{R}t_3, t_4\mathcal{R}t_4, \\ & t_5\mathcal{R}t_5, t_5\mathcal{R}t_6, t_5\mathcal{R}t_7, t_6\mathcal{R}t_5, t_6\mathcal{R}t_6, t_6\mathcal{R}t_7, t_7\mathcal{R}t_5, t_7\mathcal{R}t_7\}. \end{aligned}$$

Note that $t_7\mathcal{R}t_6$ is not a member in $\mathcal{R}$. Thus, $\{t_1, t_2, t_3, t_4\}$ is a maximal disjoint compatibility class, i.e., $\{t_1, t_2, t_3, t_4\}$ satisfies MVD(A,B,C). Therefore, relation $r(ABC)$ satisfies CWMVD(A,B=0,C).

4 Comparing Strong Versus Weak Dependencies

Our purpose in this section is show that weak dependencies are more general than strong dependencies.

Lemma 10. [2,3] MVD is a special case of WMVD.

Lemma 11. CSMVD is a special case of CWMVD.

Similarly, contextual dependencies are more general than their noncontextual counterparts.

Lemma 12. CSMVD is a more general dependency than MVD.

Lemma 13. CWMVD is a more general dependency than WMVD.

The relationships between all of these dependencies can be summarized as:

$$MVD \implies WMVD \implies CWMVD,$$

and

$$MVD \implies CSMVD \implies CWMVD.$$

It should be noted that WMVD does not logically imply CSMVD, and vice versa. For example, relation $r_2(ABC)$ in Figure 1 satisfies WMVD(A,B,C), but not CSMVD(A,B=0,C). On the other hand, relation $r_1(ABC)$ in Figure 1 satisfies CSMVD(A,B=0,C), but not WMVD(A,B,C).

5 Axiomatization of the Noncontextual Dependencies

By expressing dependencies using equivalence relations, it is straightforward to show the soundness of several inference axioms.

The following two axioms (MW1) and (MW2) are a sound and complete axiomatization for the mixture of MVD and WMVD [7]:

(MW1) If MVD(Y,X,Z), then WMVD(Y,X,Z);
(MW2) If WMVD(Y,XZ,W), WMVD(Y,XW,Z), and MVD(Z,XY,W), then WMVD(Y,X,ZW).

The soundness of axiom (MW1) follows directly from the definitions of MVD and WMVD. By definition, WMVD(Y,XZ,W), WMVD(Y,XW,Z), and MVD(Z,XY,W) imply:

$$\theta(XZY) \circ \theta(XZW) = \theta(XZW) \circ \theta(XZY), \tag{9}$$

$$\theta(XWY) \circ \theta(XZW) = \theta(XZW) \circ \theta(XWY), \tag{10}$$

and

$$\theta(XY) = \theta(XYZ) \circ \theta(XYW) = \theta(XYW) \circ \theta(XYZ), \tag{11}$$

respectively. Using Equations (9)-(11) it follows

$$\begin{aligned}
\theta(XY) \circ \theta(XZW) &= \theta(XYZ) \circ \theta(XYW) \circ \theta(XZW) \\
&= \theta(XYZ) \circ \theta(XZW) \circ \theta(XYW) \\
&= \theta(XZW) \circ \theta(XYZ) \circ \theta(XYW) \\
&= \theta(XZW) \circ \theta(XY). \qquad (12)
\end{aligned}$$

Equation (12) indicates that WMVD(Y,X,ZW) as desired.

As a second example, the following four inference axioms (W1)-(W4) are a sound and complete axiomatization for WMVD [2]:

(W1) If $U \subseteq X$, then WMVD(U,X,YZ);
(W2) If WMVD(YX,XVZ,W), then WMVD(Y,XVZ,W) and WMVD(YXV,XVZ,W);
(W3) If WMVD(Y,X,ZW), then WMVD(Y,XZ,W);
(W4) If WMVD(Y,X,Z), then WMVD(Z,X,Y).

Two properties [6] of $U \subseteq X$ are that $\theta(XU) = \theta(X)$ and

$$\theta(U) \circ \theta(X) = \theta(X) \circ \theta(U) = \theta(X).$$

Thus, inference axiom (W1) is sound since $\theta(XU)\circ\theta(XYZ) = \theta(X)\circ\theta(XYZ) = \theta(XYZ) = \theta(XYZ)\circ\theta(X) = \theta(XYZ)\circ\theta(XU)$. Therefore, $WMVD(U, X, YZ)$.

To show the soundness of (W2), we are given:

$$\theta(XVZYX) \circ \theta(XVZW) = \theta(XVZW) \circ \theta(XVZYX),$$

or equivalently,

$$\theta(XVZY) \circ \theta(XVZW) = \theta(XVZW) \circ \theta(XVZY).$$

This is the definition of WMVD(Y,XVZ,W). Now consider

$$\begin{aligned}
\theta(XVZYXV) \circ \theta(XVZW) &= \theta(XVZY) \circ \theta(XVZW) \\
&= \theta(XVZW) \circ \theta(XVZY).
\end{aligned}$$

This is the definition of WMVD(YXV,XVZ,W).

In inference axiom (W3), we are initially given:

$$\theta(XY) \circ \theta(XZW) = \theta(XZW) \circ \theta(XY).$$

We want to show:

$$\theta(XYZ) \circ \theta(XZW) = \theta(XZW) \circ \theta(XYZ).$$

Consider

$$t_1\theta(XYZ)t_2 \text{ and } t_2\theta(XZW)t_3.$$

Since $t_1(XYZ) = t_2(XYZ)$, this implies that

$$t_1\theta(XY)t_2 \text{ and } t_2\theta(XZW)t_3.$$

By the given WMVD(Y,X,ZW), we obtain

$$t_1\theta(XZW)t_4 \text{ and } t_4\theta(XY)t_3.$$

What remains to be shown is that $t_4(XYZ) = t_3(XYZ)$, namely, $t_4(Z) = t_3(Z)$. Now $t_4(Z) = t_1(Z) = t_2(Z) = t_3(Z)$. Therefore, we have our desired result:

$$t_1\theta(XZW)t_4 \text{ and } t_4\theta(XYZ)t_3.$$

The soundness of (W4) follows directly from Definition 6.

6 Conclusion

In this paper, we have suggested a new characterization of MVD and WMVD based on equivalence relations. This characterization clearly exhibits the difference between not only these two database dependencies, but also their contextual counterparts. By expressing MVD and WMVD with equivalence relations, other results can be readily shown as we demonstrated by proving the soundness of the corresponding inference axioms. More importantly, the results here can be applied to the recent interest in contextual probabilistic conditional independence in Bayesian networks.

References

1. C. Boutilier, N. Friedman, M. Goldszmidt and D. Koller, Context-specific independence in Bayesian networks, *Proceedings of the Twelfth Conference on Uncertainty in Artificial Intelligence*, 115–123, 1996.
2. P. Fischer and D. Van Gucht, Weak multivalued dependencies, *Proceedings of the Third ACM SIGACT-SIGMOD Symposium on the Principles of Database Systems*, 266–274, 1984.
3. G. Jaeschke and H.J. Schek, Remarks on the algebra on non first normal form relations, *Proceedings of the First ACM SIGACT-SIGMOD Symposium on the Principles of Database Systems*, 124–138, 1982.
4. D. Maier, *The Theory of Relational Databases*, Computer Science Press, Rockville, Maryland, 1983.
5. J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*, Morgan Kaufmann, San Francisco, California, 1988.
6. F. Preparata and R. Yeh, *Introduction to Discrete Structures*, Addison-Wesley, Don Mills, Ontario, 1973.
7. D. Van Gucht and P. Fischer, MVDs, weak MVDs and nested relational structures, *Technical report*, CS-84-19, Vanderbilt University, October, 1984.
8. S.K.M. Wong and C.J. Butz, Contextual weak independence in Bayesian networks, *Proceedings of the Fifteenth Conference on Uncertainty in Artificial Intelligence*, 670–679, 1999.