Critical Remarks on the Maximal Prime Decomposition of Bayesian Networks

Cory J. Butz, Qiang Hu, and Xue Dong Yang

Department of Computer Science, University of Regina,
Regina, Canada, S4S 0A2, {butz,huqiang,yang}@cs.uregina.ca

Abstract. We present a critical analysis of the *maximal prime decomposition* of *Bayesian networks* (BNs). Our analysis suggests that it may be more useful to transform a BN into a *hierarchical Markov network*.

1 Introduction

Very recently, it was suggested that a *Bayesian network* (BN) [3] be represented by its unique *maximal prime decomposition* (MPD) [1]. An MPD is a hierarchical structure. The root network is a *jointree* [3]. Each node in the jointree has a local network, namely, an undirected graph called a *maximal prime subgraph*. This hierarchical structure is claimed to facilitate probabilistic inference by *locally* representing independencies in the maximal prime subgraphs.

In this paper, we present a critical analysis of the MPD representation of BNs. Although the class of *parent-set* independencies is always contained within the nodes of the root jointree, we establish in Theorem 2 that this class is *never* represented in the maximal prime subgraphs (see Example 3). Furthermore, in Example 4, we show that there can be an independence in a BN defined *precisely* on the same set of variables as a node in the root jointree, yet this independence is *not* represented in the local maximal prime subgraph. Conversely, we explicitly demonstrate in Example 5 that there can be an independence holding in the local maximal prime subgraph, yet this independence cannot be *realized* using the probability tables assigned the corresponding node in the root jointree.

This paper is organized as follows. Section 2 reviews the maximal prime decomposition of BNs. We present a critique of the MPD representation in Section 3. The conclusion is presented in Section 4.

2 Maximal Prime Decomposition of Bayesian Networks

Let X, Y, Z be pairwise disjoint subsets of U. The *conditional independence* [3] of Y and Z given X is denoted $I(Y, X, Z)$. The conditional independencies encoded in the *Bayesian network* (BN) [3] in Fig. 1 on $U = \{a, b, c, d, e, f, g, h, i, j, k\}$ indicate that the joint probability distribution can be written as $p(U) = p(a) \cdot p(b) \cdot p(c|a) \cdot p(d|b) \cdot p(e|b) \cdot p(f|d, e) \cdot p(g|b) \cdot p(h|c, f) \cdot p(i|g) \cdot p(j|g, h, i) \cdot p(k|h)$.

Olesen and Madsen [1] proposed that a given BN be represented by its unique *maximal prime decomposition* (MPD). An MPD is a hierarchical structure. The

G. Wang et al. (Eds.): RSFDGrC 2003, LNAI 2639, pp. 682–685, 2003.

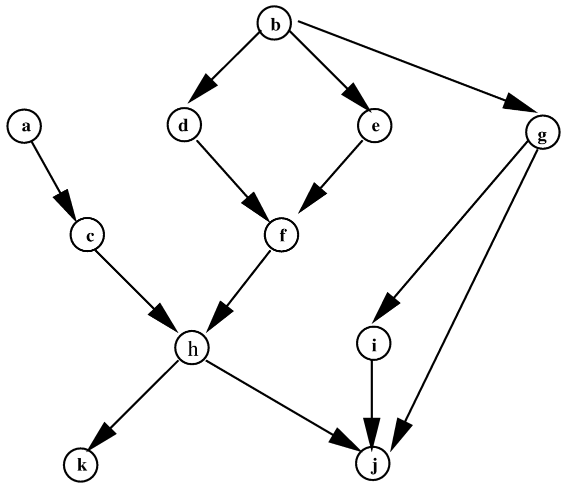

Fig. 1. A *Bayesian network* $\mathcal{D}$ encoding independencies on the set U of variables.

root network is a *jointree* [3]. Each node in the jointree has a local network, namely, an undirected graph called a *maximal prime subgraph.*

Example 1. Given the BN $\mathcal{D}$ in Fig. 1, the MPD representation is shown in Fig. 2. Each of the five nodes in the root jointree has an associated maximal prime subgraph as denoted with an arrow.

The root jointree encodes independencies on U, while the maximal prime subgraphs encode independencies on *proper* subsets of U. For example, the root jointree in Fig. 2 encodes $I(a, c, bdefghijk)$, $I(abcdefgij, h, k)$ $I(ack, fh, bdegij)$, and $I(abcdefk, gh, ij)$ on U, while $I(bde, fg, h)$, for instance, can be inferred from the maximal prime subgraph for node $bdefgh$.

The numerical component of the MPD is defined by assigning the conditional probability tables of the BN to the nodes of the jointree. In our example, this assignment must be as follows: $\phi_1(ac) = p(a) \cdot p(c|a)$, $\phi_2(cfh) = p(h|c, f)$, $\phi_3(bdefgh) = p(b) \cdot p(d|b) \cdot p(e|b) \cdot p(f|d, e) \cdot p(g|b)$, $\phi_4(ghij) = p(i|g) \cdot p(j|g, h, i)$, and $\phi_5(hk) = p(k|h)$.

3 Critical Remarks on the MPD of BNs

A *parent-set* independency $I(Y, X, Z)$ is one such that YXZ is the *parent-set* [2] of a variable a_i in a BN. For example, the BN $\mathcal{D}$ in Fig. 1 encodes the parent-set independencies $I(c, \emptyset, f)$ and $I(h, g, i)$; c and f are the parents of variable h, while g, h and i are the parents of j. The proof of the next result is omitted due to lack of space.

Theorem 2. All parent-set independencies in a Bayesian network are *not* represented in the maximal prime decomposition.

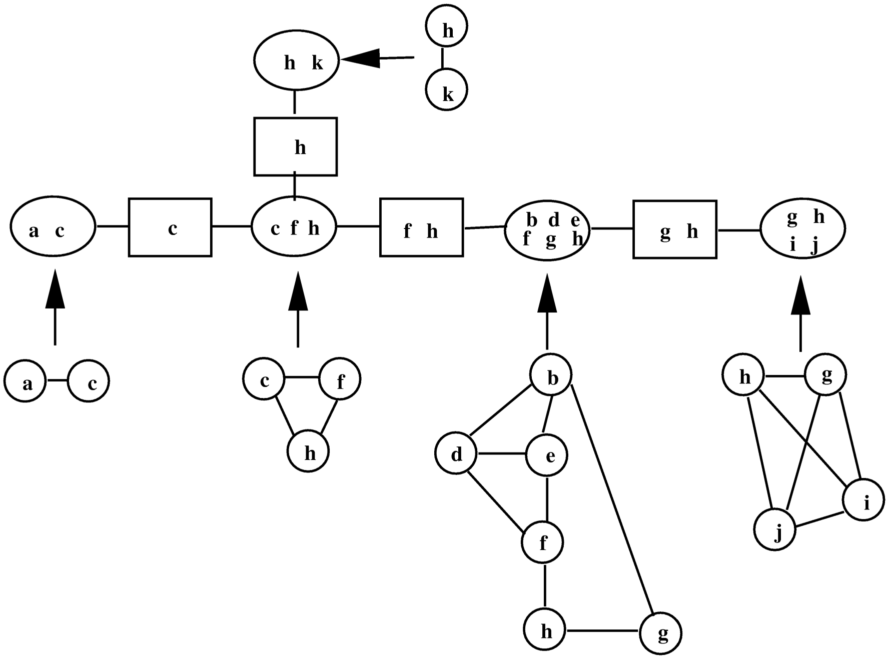

Fig. 2. The *maximal prime decomposition* (MPD) of the BN $\mathcal{D}$ in Fig. 1.

Example 3. Although the BN $\mathcal{D}$ in Fig. 1 indicates that variables c and f are *unconditionally independent*, the maximal prime decomposition of D indicates that c and f are *dependent*. Similar remarks hold for the parent-set independence $I(h, g, i)$.

Example 4. $I(defh, b, g)$ holds in the given BN. On the contrary, b does *not* separate $\{d, e, f, h\}$ from g in the maximal prime subgraph $bdefgh$ as g and h are directly connected (dependent).

Example 5. $I(bde, fg, h)$ can be inferred by separation from the maximal prime subgraph $bdefgh$. However, it can *never* be realized in the probability table $\phi(bdefgh)$ as variable h does *not* appear in any of the conditional probability tables $p(b)$, $p(d|b)$, $p(e|b)$, $p(f|d, e)$, $p(g|b)$ assigned to $\phi(bdefgh)$.

In [3], Wong et al. suggested that a BN be transformed in a *hierarchical Markov network* (HMN). An HMN is a hierarchy of Markov networks (jointrees). The primary advantages of HMNs are that they are a *unique* and *equivalent* representation of BNs [3]. For example, given the BN $\mathcal{D}$ in Fig. 1, the *unique* and *equivalent* HMN is depicted in Fig. 3.

Example 6. The BN $\mathcal{D}$ in Fig. 1 can be transformed into the unique MPD in Fig. 2 or into the unique HMN in Fig. 3. Unlike the MPD representation

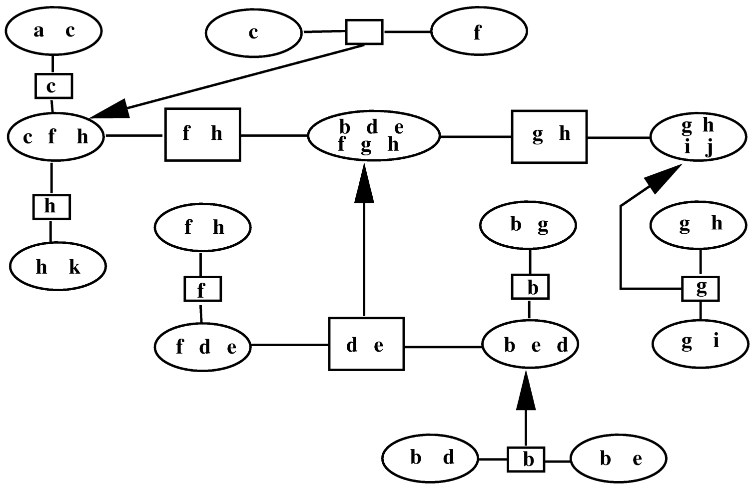

Fig. 3. The *hierarchical Markov network* (HMN) for the BN $\mathcal{D}$ in Fig. 1.

which does *not* encode, for instance, the independencies $I(c, \emptyset, f)$, $I(h, g, i)$, and $I(d, b, e)$, these independencies are *indeed* encoded in the appropriate nested jointree of the HMN.

4 Conclusion

The *maximal prime decomposition* (MPD) [1] of *Bayesian networks* (BNs) is a very limited hierarchical representation as it always consists of precisely two levels. Moreover, the MPD is undesirable since it is *not* a faithful representation of BNs. On the contrary, it has been previously suggested that BNs be transformed into *hierarchical Markov networks* (HMNs) [3]. The primary advantages of HMNs are that they are a *unique* and *equivalent* representation of BNs [3]. These observations suggest that compared with the MPD of BNs, HMNs seem to be a more desirable representation.

References

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