Pattern Recognition based on Lie Group Covariance Feature

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Abstract. This paper analyzes images' Lie group covariance features, based on the main idea of [8], we propose LLDA (Lie group Linear Discriminant Analysis) algorithm for image classification. The main idea of this algorithm is to apply LDA to images' covariances, which forms a Lie group manifold, and compute a one-parameter sub group determined by a Lie algebra element and the intrinsic mean of image features. This one-parameter sub group is a geodesic on the Lie group formed by original image set. By defining the projection in Lie group, this geodesic can be calculated by the idea of LDA. Experimental results on handwritten classification show that LLDA has significantly better classification performance than some classic methods such as LDA (linear discriminant analysis).

Keywords: Pattern recognition, covariance feature, Lie group, LDA

1 Introduction

Pattern recognition is one of the most important applications in image processing [4]. Image features are extracted to compute similarity or dissimilarity among images. Most kinds of features are in vector form. Therefore, based on vector geometry, it is easy to derive metrics such as the Euclidean distance [5], Mahalanobis distance [10], Chebyshev distance [3], or Manhattan distance [9]. Further, algorithms (e.g., K-nn) based on these distances will be adopted to do recognition. However, some image features are not in vector form, they are even not distributed in euclidean space, this means we cannot use common metrics to compute the similarity between images.

Image's covariance feature [15] is one of image features that is in matrix form. It has many advantages such as easy to extract from images and with very few dimensions. Therefore, it is widely used in pattern recognition [15], object tracing [12], and object detection [6]. However, the space formed by this feature is not a euclidean space, even not linear space. Thus, the distance between covariance matrices cannot be directly calculated by metrics derived in euclidean space. Actually, the covariance features form a symmetric semi-positive matrix group [1], which is a manifold space, more general than euclidean space and similarity formula between images has to be derived in manifold.

By adding an operator to covariance matrices, this symmetric semi-positive matrix group further constructs a Lie group [1]. Lie group is a special group since it is not only an algebraic group but also a differential manifold. Therefore, it has differential geometric structure, and we can also use algebriac method to manipulate it.

The purpose of this paper is to make an analysis of covariance features and to propose an pattern recognition algorithm. We first briefly review covariance image features and Lie group theory, and based on the analysis of these fundamentals, then an algorithm called LLDA is derived based on the principle of LDA. The rest content is organized as follows: Section 2 reviews covariance features and Lie group theory. Section 3 introduces LLDA, the algorithm based on LDA. Section 4 provides some experimental results of LLDA and some comparison with other common algorithms are also given. Section 5 is conclusion.

2 Covariance Feature and Its Lie Group Space

In this section, an introduction of covariance feature is given and an analysis of some operations in its formed Lie group manifold space are shown.

2.1 The Covariance Feature

A digital image with resolution $M \times N$, is a set of values $I(u, v)$, in which $u = 0, \cdots, N-1$ and $v = 0, \cdots, M-1$ are coordinates, $I(u, v)$ is the intensity at the coordinate row u and column v. Some operations on image can be derived, such as first derivative images and second derivative images. Covariance feature of image proposed by Tuzel et. al. is a statistics on a set of these values. By given a set of vectors $\{\phi(u, v)\}_{u=0,\cdots,M-1;v=0,\cdots,N-1}$, the covariance matrix can be calculated by:

$$C_R = \frac{1}{M * N - 1} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} (\phi(u, v) - \mu)(\phi(u, v) - \mu)^T, \tag{1}$$

where μ is the mean of $\{\phi(u, v)\}_{u=0,\cdots,M-1;v=0,\cdots,N-1}$.

2.2 Distance and Mean of Covariance Feature

By extracting covariance feature, each image can be represented as a covariance matrix. Therefore, pattern recognition can be done among covariance matrices with some operations.

First, any covariance matrix is a symmetric positive-semidefinite matrix and all covariance matrices with the same dimension of k form a symmetric positive-semidefinite matrix group $Sym^+(k)$ [2]. Pennec et. al. [11] provide a detail theoretic analysis of this group. Further, Arsigny et. al. [1] proved that $Sym^+(k)$ will be a Lie group $G = (Sym^+(k), \odot, ^{-1})$ by adding an operator $\odot$ as group multiplication operation defined in Equation (2) to this group:

$$\forall z_1, z_2 \in Sym^+(k), \quad z_1 \odot z_2 = exp(log(z_1) + log(z_2)), \tag{2}$$

where $log(\cdot)$ and $exp(\cdot)$ are matrix logarithm and exponential.

Based on Arsigny et. al.'s research [1], distance between any two covariance matrix z_1 and z_2 can be computed by $d(z_1, z_2)$:

$$\begin{aligned} d(z_1, z_2) &= \| log(z_1^{-1} \odot z_2) \|_F \\ &= \| log(z_1) - log(z_2) \|_F, \end{aligned} \tag{3}$$

where $\| \cdot \|_F$ is Frobenius norm. The intrinsic mean of a set of covariance matrices can be computed by:

$$\mu = exp\big(\frac{1}{n}\sum_{i=1}^{n} log(z_i)\big). \tag{4}$$

3 Lie Group Linear Discriminant Analysis for Pattern Recognition

Based on distance and mean in Lie group, we can adopt apply the main idea of LDA [7] to find a geodesic in Lie group for classification.

3.1 Projection in Covariance Lie Group

The Fisher projection [7] tries to project a set of points to another space with less dimensions. In the new space, the inter-class variance is minimized together with the intra-class variance is maximized.

Because the space formed by covariance matrices is a Lie group manifold, which is not a linear space, thus, we try to find a geodesic on this manifold to classify different categories of images via their covariance features.

According to the Lie algebra $\mathfrak{g}$ associated with Lie group G, in which $\mathfrak{g}$ is the tangent space on the identity element $E \in G$ and by given a direction on this tangent space, i.e., $v \in \mathfrak{g}$, we can generate a geodesic on G by going through this direction. In other words, a geodesic H_v is a one-parameter subgroup of Lie group G via a one-parameter exponential mapping, $R \to G$:

$$H_v \triangleq \{exp(tv) \in G,\ t \in R\}, \tag{5}$$

such that any point on this geodesic can be determined by a specific value t. To generalize the concept of projection to manifold, we can find the projection of a point $z \in G$ on the geodesic H_v by calculating the t^*, which satisfies the following condition:

$$t^* = \arg\min_{t \in R} d(z, exp(tv)). \tag{6}$$

Then, $exp(t^* v)$ is the projection of z on H_v. The geometrical meaning is that z's projection on H_v is a point on this geodesic and the distance between z and its projection is the shortest.

According to the projection defined in Equation (6), a set of points $\{z_i\}_{i=1,\cdots,n}$ in Lie group can be projected onto a geodesic and therefore transformed to a corresponding set of projections $\{\tilde{z}_i\}_{i=1,\cdots,n}$. However, we do not have to calculate the real value t for each instance in Lie group since Lie algebra is a linear vector space, the projection of a given $z \in G$ on direction v ($||v||_F = 1$) in Lie algebra can be computed by $v^T log(z)$, therefore the projection in G can be obtained by:

$$\tilde{z} = exp(v^T log(z)). \tag{7}$$

Equation (7) eliminates the parameter t and is easier to compute the corresponding projection point.

3.2 Lie Group Linear Discriminant Analysis

As matrix Lie group, the inverse operation is the matrix inverse operation, i.e., the inverse of an element in the group is its inverse matrix. This inverse operation can be used to remove one element's component from another element. In other words, if we want to get the difference of two Lie group elements $z_1, z_2 \in G$, compute $z_1^{-1} \odot z_2$, i.e., a matrix inverse operation and a matrix multiplication operation. This is done by left translation $L_{z_1^{-1}}(z_2) = z_1^{-1} \odot z_2$. Therefore, S_b and S_w in LDA can be computed from:

$$
\begin{aligned}
S_b &= \sum_{i=1}^{c} n_i \log(\mu^{-1} \odot \mu_i) \log(\mu^{-1} \odot \mu_i)^T \\
&= \sum_{i=1}^{c} n_i (\log(\mu^{-1}) + \log(\mu_i))(\log(\mu^{-1}) + \log(\mu_i))^T, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
S_w &= \sum_{i=1}^{c} \sum_{j=1}^{n_i} \log(\mu_i^{-1} \odot z_{ij}) \log(\mu_i^{-1} \odot z_{ij})^T \\
&= \sum_{i=1}^{c} \sum_{j=1}^{n_i} (\log(\mu_i^{-1}) + \log(z_{ij}))(\log(\mu_i^{-1}) + \log(z_{ij}))^T. \qquad (9)
\end{aligned}
$$

Therefore, the objective function turns to:

$$
J(v) = \frac{v^T \sum_{i=1}^{c} n_i log(\mu^{-1} \odot \mu_i) log(\mu^{-1} \odot \mu_i)^T v}{v^T \sum_{i=1}^{c} \sum_{j=1}^{n_i} log(\mu_i^{-1} \odot z_{ij}) log(\mu_i^{-1} \odot z_{ij})^T v}. \qquad (10)
$$

According to LDA, v can be obtained by solving $S_b v = \lambda S_w v$, which is an eigenvalue decomposition problem. When $c = 2$, the projective direction v can be obtained by:

$$
\begin{aligned}
v &= S_w^{-1} log(\mu_1^{-1} \odot \mu_2) \\
&= \left(\sum_{i=1}^{2} \sum_{j=1}^{n_i} log(\mu_i^{-1} \odot z_{ij}) log(\mu_i^{-1} \odot z_{ij})^T \right)^{-1} log(\mu_1^{-1} \odot \mu_2) \\
&= \left(\sum_{i=1}^{2} \sum_{j=1}^{n_i} (log(z_{ij}) - log(\mu_i))(log(z_{ij}) - log(\mu_i))^T \right) \cdot (log(\mu_2) - log(\mu_1)). \qquad (11)
\end{aligned}
$$

Consequently, the Lie group Linear Discriminant Analysis algorithm can be easily devised in Algorithm 1. The time complexity of Algorithm 1 is $O(n)$.

3.3 Pattern Recognition based on LLDA

After projecting all instances on to a geodesic, instances belonging to different classes are easy to classify. Given an unlabelled instance, we can first project it onto the geodesic found in Algorithm 1, then compare the distances between this projection and each projection of classes' means on this geodesic, the class label will be assigned to the label owned by the shortest mean

Algorithm 1: The Lie group Linear Discriminant Analysis (LLDA) algorithm

input : A set of covariance feature of images $\{z_{ij}\}_{i=1,\cdots,c;j=1,\cdots,n_i}$, where z_{ij} is j^{th} covariance feature in i^{th} class and there are c classes in total.

output: The projective direction v of geodesic H_v.

(1) Compute the the mean μ of whole set and means μ_i for each class, $i = 1, \cdots, c$ by Equation (4);

(2) Compute the S_b and S_w by Equation (8) and (9);

(3) Obtain v by solving equation $S_b v = \lambda S_w v$ or Equation (11);

(4) Return v.

to the instance's projection. Formally, the class label can be computed by following equation:

$$\begin{aligned} i^* &= \arg \min_{i=1,\cdots,c} d(\tilde{z}, \tilde{\mu_i}) \\ &= \arg \min_{i=1,\cdots,c} \parallel v^T(log(z) - log(\mu_i)) \parallel_F . \end{aligned} \tag{12}$$

Intuitively, this equation tells us that we can first compute the vector from point μ_i to z, then project this vector on geodesic H_v, finally compute the length of this vector.

4 Experimental Results

In this section, we show some experimental results of LLDA on MNIST handwritten data set [16]. The MNIST data set collects images of digital number from 0 to 9, each image has resolution of 28×28 and 256 grey levels. In the following experiments, we use following vectors to compute the covariance feature for each image:

$$\phi_1(u,v) = (u, v, I(u,v), |\partial I(u,v)/\partial u|, |\partial I(u,v)/\partial v|) \tag{13}$$

$$\phi_2(u,v) = (\phi_1(u,v), (|\partial I(u,v)/\partial u|^2 + |\partial I(u,v)/\partial v|^2)^{1/2}) \tag{14}$$

$$\phi_3(u,v) = (\phi_1(u,v), |\partial^2 I(u,v)/\partial u^2|^2, |\partial^2 I(u,v)/\partial v^2|^2) \tag{15}$$

$$\phi_4(u,v) = (\phi_3(u,v), tan^{-1}\big(|(\partial I(u,v)/\partial v)/(\partial I(u,v)/\partial u)|\big)) \tag{16}$$

The first experiment is to distinguish numbers 3 and 5. To train the LLDA, it randomly chooses 100 images from training set for number 3 and 5, respectively. To test the algorithm, it randomly chooses 200 images from testing set for both classes. The experiment is run for 10 times and the results are shown in Fig. 1, where algorithm LieMean assigns the class label whose mean is closest to the testing instance and algorithm LDA is the classic linear discriminant analysis method with the same classification criteria and linear operations (e.g., extrinsic mean) without Lie group operations.

The second experiment is to distinguish numbers 1 and 9 and the same setting as previous is used. The results are shown in Fig. 2, it is obvious that LLDA has the best performance in all cases (with $k = 5$ to 8) and the performance has a slight growth when k increases. In distinguishing 1 and 9, LLDA has nearly 99% recognition rate. The LieMean is not stable in two experiments due to the distribution of covariance feature in the manifold, while LDA is stabler than LieMean.

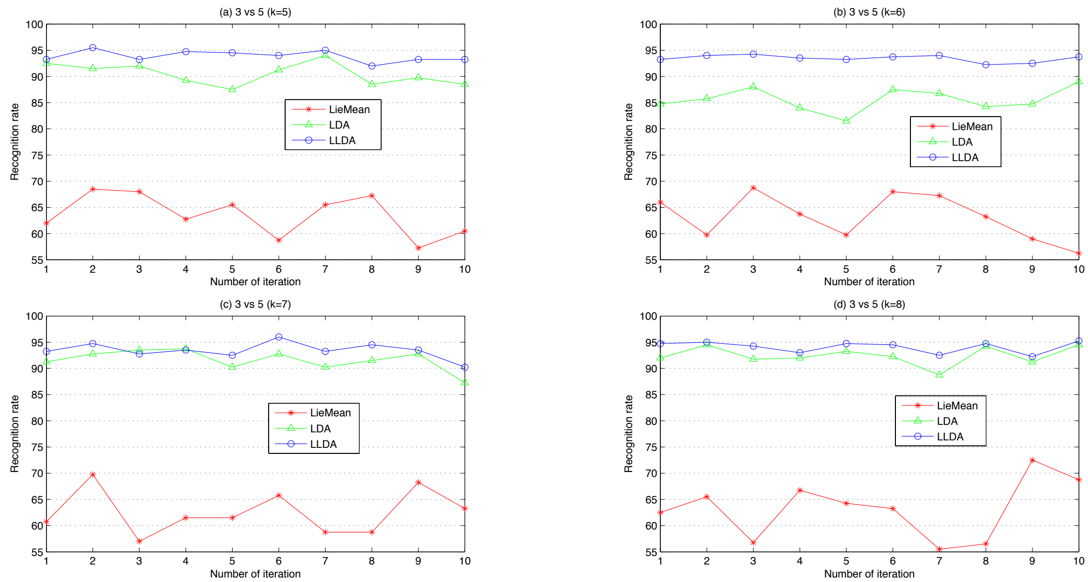

Fig. 1. (a) - (d) are recognitions between handwritten numbers 3 and 5, $k = 5$ to 8 correspond to Equation (13) to (16), respectively.

Experiments also show that although the covariance features are in Lie group, the formed space is much linear due to LDA also has a good performance.

As for the computation time, LLDA has $O(n)$ complexity as analyzed above and each iteration in the experiment is finished within one second on an equipment of Intel i5 CPU with 2 cores at 2.4GHz, 8GB RAM, Mac OS X 10.9 and Matlab 2013a.

5 Conclusion

Pattern recognition algorithms play an important role in image processing. When the original image set consists of different resolutions or in the real-time scenario, covariance feature will be a good choice.

An important criterion in designing algorithm is to make the operation reasonable to the space formed by the data. Based on the Lie group manifold space formed by covariance feature, we derived LLDA according to the basic idea of linear discriminant analysis. Through experiments on handwritten classification, results show that LLDA has obvious best performance among the algorithms.

As future work, we will examine the performance of LLDA on multi-class recognition. Based on the Lie group operation of covariance feature, other algorithm such as clustering may be derived. Additionally, the motion data in computer vision such as affinity transforms are also Lie group [14], the proposed method LLDA may be adopted in those applications.

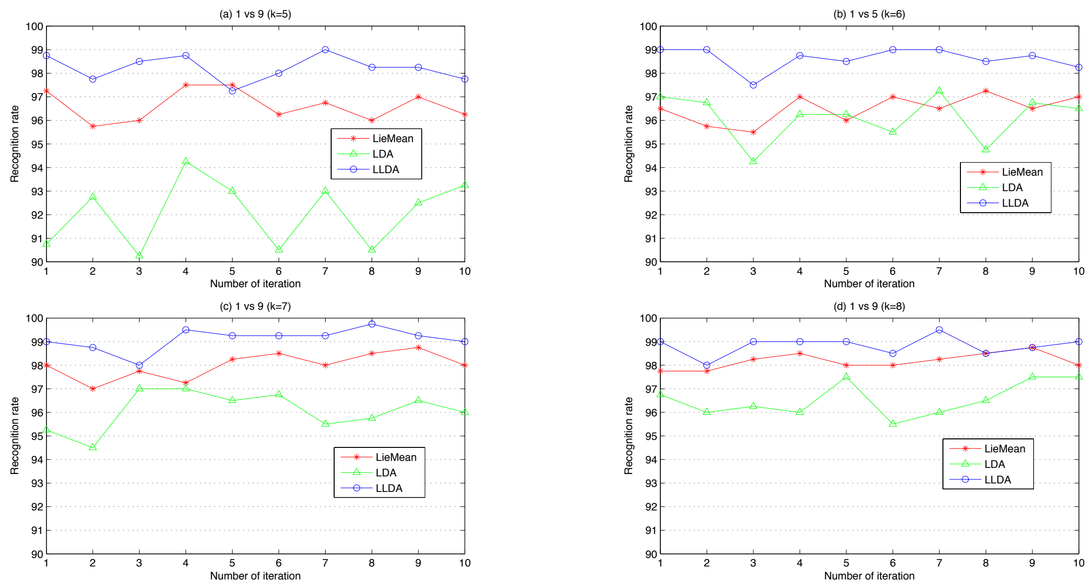

Fig. 2. (a) - (d) are recognitions between handwritten numbers 1 and 9, $k = 5$ to 8 are corresponding to Equation (13) to (16), respectively.

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