

# Mapping Tone Helixes to Cylindrical Lattices using Chiral Angles

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## ABSTRACT

*The concept of a tone helix has been studied in tone theory and harmonic analysis from a variety of different perspectives. A tone helix represents harmonic relationships between tones in an attempt to model the perception of pitch and harmony in a single form. This paper presents a framework whereby previous helical tone representations can be considered together as one generalization with multiple instantiations. The framework is realized by combining the concept of isomorphic note layouts with cylindrical lattices. The extensively studied geometry of carbon nanotubes is used as a mathematical grounding. Existing tone helix representations are shown to adhere to this new, more general framework, and a process for mapping any flat isomorphism to its corresponding tone helix is presented.*

## 1. INTRODUCTION

Euler's tonnetz [1] is perhaps the earliest exploration of the harmonic arrangement of tones on a lattice. Euler sought to build a representation which showed that notes in a scale are related not just to adjacent notes, but also (or perhaps more so) to notes which share a harmonic relationship. Perceptually, a perfect 5<sup>th</sup> with frequency ratios of 3:2 can be considered a closer relationship than that of a semitone.

Music theory has long since encapsulated this concept with the *circle of fifths* which shows close relationships between keys. Researchers have also explored 3-dimensional helical structures of pitch, showing the harmonic relationships between intervals as the overall pitch ascends. Recently, research into isomorphic layouts has shown a generalization from the Tonnetz, and other alternative layouts such as the Jankó, into a theory that presents any and all such harmonically related layouts, in either a square or hexagonally tiled surface [2]. We term these layouts as *flat isomorphisms*.

The purpose of this paper is to merge these two research areas, making use of ideas from the study of flat isomorphisms to further the exploration of the helical nature of musical harmony. Although all of the discussions herein and most historical explorations of these harmonic relationships have con-

centrated on the 12-tone equal tempered scale of the Western classical music tradition, it should be noted that any scale which has harmonic frequency ratios at the core can be similarly explored, including microtonal scales.

## 2. SPIRAL MODELS OF RELATIVE PITCH

Musicians and composers have always been interested in the intricate way in which humans perceive the relationship between pitches. Physical constraints mean that most musical instruments align pitch on a linear scale, with adjacent notes being close together in frequency. For a western 12-tone equal tempered scale, this means that adjacent notes are a semitone apart, but the semitone is not a harmonically consonant interval. Intervals with small whole-number frequency ratios, such as the perfect 5<sup>th</sup> (3:2), the perfect 4<sup>th</sup> (4:3) and the major 3<sup>rd</sup> (5:4) have the feeling of harmonic closeness, and researchers have explored the possibility of a representational structure that showed these harmonic relationships instead of (or in addition to) the frequency relationship. Repeating ascending octaves imply that these harmonic relationships are helical, which is why many researchers have independently investigated pitch spirals or harmonic helixes.

Drobisch originally proposed the idea that pitch height could be represented as a helix, in 1855 [3]. In 1982, Shepard [4] introduced an equal-spaced helical model to arrange twelve chromatic pitches over a regular, symmetrical, transformation-invariant surface. Shepard notes that this model could be isomorphic, and allows a differential stretching or shrinking of the vertical extent of an octave of the helix relative to its diameter. Shepard's spiral pitch model is shown in Fig. 1a.

Krumhansl [5] tried to use empirical data to unveil the relationships of pitches in tonality. She proposed a conical structure of pitch intervals which corroborates the perceptual neo-Riemannian transformation, and does not contradict Shepard's spiral model. Both Shepard and Krumhansl's models are based on the psychological perception of pitch, but since they are both abstracted structures based on a single octave, they shown no information on pitch relationships beyond the octave. In both of these models, the position of a pitch is related using height  $h$  and radius  $r$ , providing an angle of the helix itself from the plane as a ratio of  $h/r$ .

Based on those two structures and the Longuet-Higgins's shape match algorithm, Chew [6] explored an abstract spiral model for mapping Tonnetz-based representations to the helix, providing an identical distance between each perfect 5<sup>th</sup>

interval, and a different identical distance between each major/minor 3<sup>rd</sup>. This arrangement is equivalent to a specific isomorphism wrapped into a cylinder, using 5<sup>th</sup>s and 3<sup>rd</sup>s as the defining intervals. In Section 5, we will show that Chew’s pitch representation, when appropriately constrained, corresponds to one case of a family of similar pitch models, which can be enumerated using our proposed framework. Chew’s abstract spiral model model is shown in Fig. 1b.

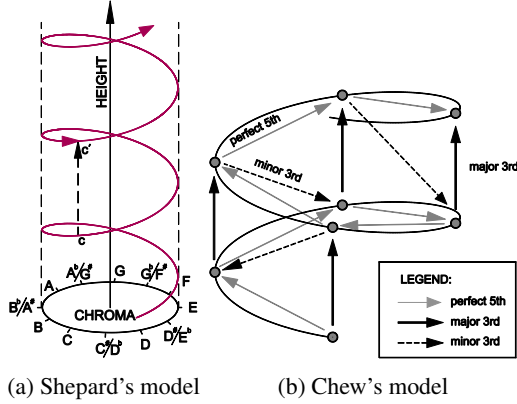


Figure 1: Helical models of pitch.

### 3. HEXAGONAL ISOMORPHIC LAYOUTS

An *isomorphic layout* is an arrangement of pitches such that any musical construct (scale, chord, melody) has a consistent shape regardless of the root pitch of the construct. Early examples of isomorphic layouts include Euler’s tonnetz and Jankó’s keyboard [7], and recent musical instruments based on isomorphisms include the C-thru Axis and the Opal keyboard. A general theory of isomorphisms [2] states that, given any two intervals, an isomorphic layout can be constructed and evaluated for completeness (*i.e.* that it contains all notes in the given musical system). As with the rest of this discussion, isomorphic layouts are not restricted to the western 12-tone equal tempered scale, but we use this scale and nomenclature in our discussion for convenience and familiarity.

Isomorphisms have in the past been limited to flat surfaces, however, 3d geometries for hexagonal lattices provide a compelling opportunity for isomorphic study. The next section presents a discussion of the mathematics of *cylindrical* hexagonal lattices, using the mathematical context developed in the study of *carbon nanotubes*.

### 4. CYLINDRICAL HEXAGONAL LATTICES

A chiral tube  $(n, m)$ , is defined by a *chiral vector*  $\vec{Ch}$ , indicating the orientation of the hexagonal lattice on the tube:

$$\vec{Ch} = n \cdot \vec{a1} + m \cdot \vec{a2}, \quad (1)$$

where  $\vec{a1}$  and  $\vec{a2}$  are two basis vectors separated by  $60^\circ$ .

We can imagine cutting a planar hexagonal lattice in a specific direction, along the edges of hexagons, and then curling

the resulting sheet into a cylinder. If we cut along the chiral angle of  $0^\circ$ , we get a special tube known as a “zigzag”. Cutting along the chiral angle of  $30^\circ$  gives us a the “armchair” tube. Any other angle between  $0^\circ$  and  $30^\circ$  gives a general chiral tube. These three different cutting directions are shown in Fig. 2, and the resulting tubes are shown in Fig 3.

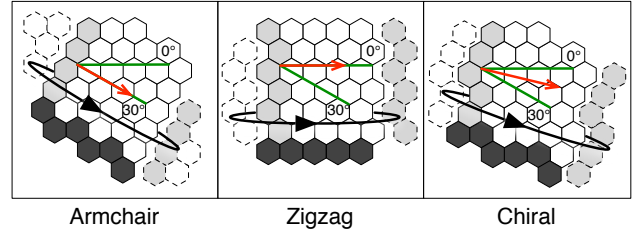


Figure 2: Three types of hexagon lattice cuttings. Dark grey indicates the “end” of the resulting tube, and light grey indicates the “seam” of the tube.

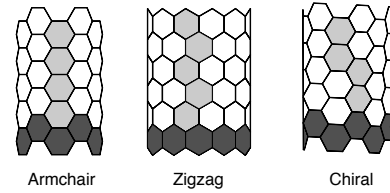


Figure 3: Three types of cylindrical hexagonal tubes, generated by cutting the planar hexagonal lattice as in Fig. 2.

The diameter of the resulting tube depends on the number of hexagons along the chiral angle. For armchair and zigzag tubes, one can create a tube of any number of hexagons, but for general chiral tubes we must select a whole number multiple of the *length* of the chiral vector. The chiral vector shows when the tube will repeat, and itself represents a whole number of hexagons in the  $\vec{a1}$  and  $\vec{a2}$  directions. Multiples of the chiral vector gives duplicates of the cutting around the tube.

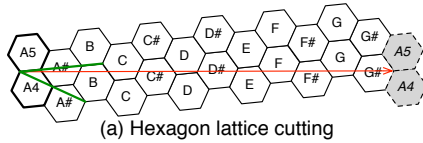
### 5. CHIRAL TUBES AND HELICAL MODELS

If we replace the hexagons on a chiral tube with individual tones, we can see the beginnings of a tone helix model appear. As we proceed around the tube, each adjacent hex corresponds to a specified interval, and the tones spiral around the tube in exactly the same way as any of the the pitch helix models presented in Section 2 would dictate.

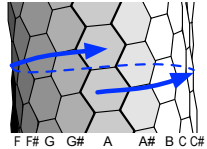
#### 5.1 Shepard’s Model

Let us first consider the Shepard pitch helix. In this case, the pitch increases by semitones around the spiral, completing one turn of the spiral once per octave. If we map this onto a hexagon tube, it means that in order to advance one octave (one hexagon *along* the tube),  $(\vec{a1})$ , we must first proceed 12 semitones (hexagons) *around* the tube  $(\vec{a2})$ . We can then calculate the angle of the chiral vector [8]:

$$\Theta = \tan^{-1} \left[ \frac{\sqrt{3m}}{m+2n} \right] = \tan^{-1} \left[ \frac{\sqrt{3 \cdot 12}}{12+2} \right] = 23.2^\circ \quad (2)$$



(a) Hexagon lattice cutting



(b) Resulting chiral tube

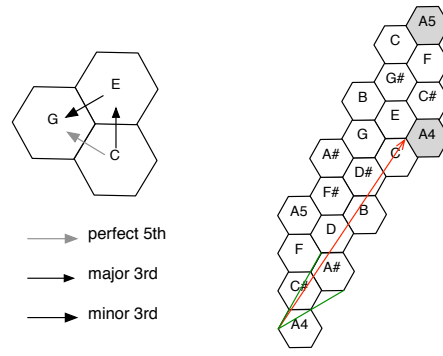
Figure 4: Chiral tube version of Shepard’s tone helix.

The hexagon lattice cutting for this chiral angle, and the resulting tube, are shown in Fig. 4. In this case, we have adjacent hexagons in one direction corresponding to semitones, and adjacent hexagons in the other direction corresponding to octaves. Shepard’s model allows for varying distance between the loops of the spiral, and we can accomplish this by allowing duplicates of the hexagon lattice cutting, resulting in a larger-diameter tube.

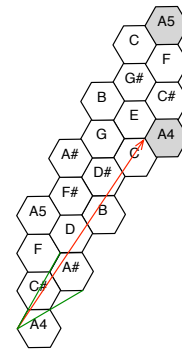
## 5.2 Chew’s Model

This method can also be used to instantiate Chew’s helical model, presented in Fig. 1b. We can see that vertically adjacent hexagons should be a major 3<sup>rd</sup> apart, and hexagons along the spiral should be a perfect 5<sup>th</sup> apart, as shown in Fig. 5. This presents a problem, however, because the chiral vector for this arrangement of notes is not circumferential to the resulting tube. This means that if we were to actually construct Chew’s model, with notes equally spaced out, it would not be internally consistent. If you start at one note and proceed around the circumference of the cylinder defined by Chew’s model, you would not get back to the same note again, leading to a paradox. Chew [6] acknowledges that distances on her helical model do not correspond to musical distance. Our framework adds a more rigorous constraint that all tones must be equidistant along and between each helix.

We can make a small modification (shown in Fig. 6) in order to make Chew’s original model consistent using equal distances. We rotate and mirror the model so that major 3<sup>rds</sup> are along the spiral, and perfect 5<sup>ths</sup> are in the vertical direction. In this way, we can make the chiral angle horizontal, as is required in our framework (see Section 6). It may also be possible to implement Chew’s original pitch helix by allowing additional notes to appear between each note on the helix, and then removing or ignoring the interspaced notes. This is left for future work.



(a) Chew’s model on a hexagonal lattice



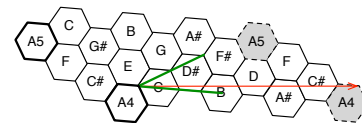
(b) Cutting required to implement Chew’s model

Figure 5: Chew’s original model cannot be implemented with fixed note size. The chiral angle is not horizontal.

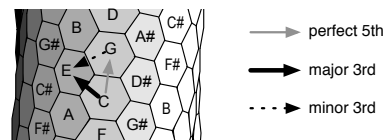
## 5.3 The Generalized Case: Finding the Helix Angle

We can imagine a tone helix with any interval along the spiral and any other interval between spiral loops, and the result can be mapped around a tube. If two intervals are sufficient to define such a tube, and likewise two intervals are sufficient to define a planar hexagonal isomorphism, then it follows that if we take any planar hexagonal isomorphism, cut it in a specific way, and wrap it around a tube, the result (if the correct cutting is chosen) will be a chiral tube corresponding to the original isomorphism. We call this a *cylindrical hexagonal isomorphism*. In this way, any unique self-consistent pitch spiral model is an instance of our generalized framework, and the helix angle for the corresponding tone spiral is, indeed, the chiral angle of the matching tube.

The primary contribution of this new framework, then, is a mathematical model of the *shear* required to represent a specific tone helix. Tube-like lattices have been proposed in the past, but researchers have only speculated as to the angle that a specific helix would need, and the circumference and shear of the corresponding lattice. The next section presents the justification for using chiral angles to compute these values for any given tone helix.



(a) Hexagon lattice cutting



(b) Resulting chiral tube

Figure 6: Chiral tube version of a modified Chew tone helix.

## 6. ISOMORPHISMS AND CHIRAL ANGLES

Each planar isomorphism, defined by two musical intervals, can be represented by a pitch axis and an isotone axis [2]. The pitch axis represents the direction in which pitches increase, and the isotone axis (orthogonal to the pitch axis) represents the direction in which pitches repeat. Adjacent isotone axes are a semitone apart, though semitones may not be adjacent on the layout. We can build on this isomorphism framework to map any isomorphism onto an appropriate chiral tube.

If you proceed around the circumference of a chiral tube created by curling around the circumference of a chiral tube, you must eventually arrive back at the original pitch on which you started. For this reason, we can see that the isotone axis of an isomorphism must be aligned with the circumference of the associated chiral tube. Since the pitch axis is orthogonal to the isotone axis, the pitch axis must therefore be aligned along the axis of the chiral tube. The chiral angle of a tube corresponds to the direction in which we must cut the hexagonal lattice to form the end of the tube. The chiral angle is therefore aligned with the circumference of the tube, meaning the chiral angle must be equal in both direction and magnitude to the isotone axis.

If we take an example isomorphism, where adjacent hexagons have major 3<sup>rd</sup>s, minor 3<sup>rd</sup>s, and semitones, we can see this process. The pitch axis and isotone axis for this isomorphism are shown in Fig. 7.

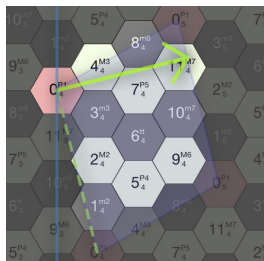


Figure 7: Pitch axis (solid arrow) and isotone (dotted line).

We can then generate a paper prototype of a chiral tube for this isomorphism (Fig. 8). This layout corresponds to a tone helix with semitones along the spiral and major 3<sup>rd</sup>s from one loop to the next, with three loops making an octave. This tube consists of one instance of the hexagonal lattice cutting. Allowing duplicates of the cutting results in a larger tube.

## 7. CONCLUSION

Isomorphic tone layouts have been popular for the exploration of musical harmony, the expanding of compositional possibilities, and the accelerated learning of musicianship and performance skill. Tone spirals have been a popular way to study and relate the intricate way humans perceive musical harmony. Combining these fields to create a class of cylindrical isomorphisms has the potential to further expose harmonic structure and offer new ways to interact with tone maps.

This paper presents a framework for mapping any tone helixes onto a cylindrical lattice by calculating the required an-

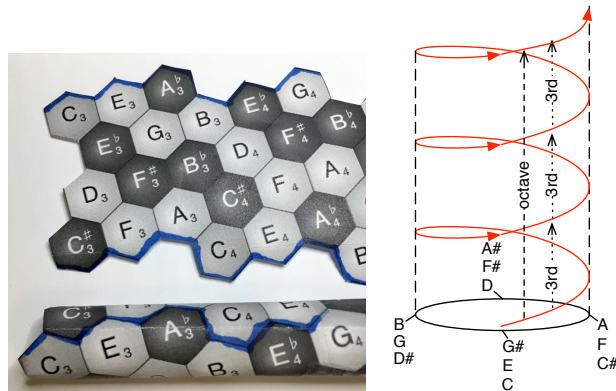


Figure 8: Low-Fi prototype tube and corresponding tone helix for the isomorphism in Fig. 7.

gle and shear, and by doing so we have shown that existing helical pitch models are instances of this framework, and that any isomorphism can be mapped to a corresponding tube by matching the isotone axis to the chiral angle. The strict requirement of fixed note distance means that some existing models must be slightly modified, but this constraint leads to more well-defined helical tone spaces.

We suspect that these cylindrical models may provide playability and composition opportunities just as flat isomorphic keyboards have done. We plan to study the chiral tubes of different tone spirals, and to construct physical instruments to evaluate the musicality of such isomorphic tubes.

## 8. REFERENCES

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