Mapping Tone Helixes to Cylindrical Lattices using Chiral Angles

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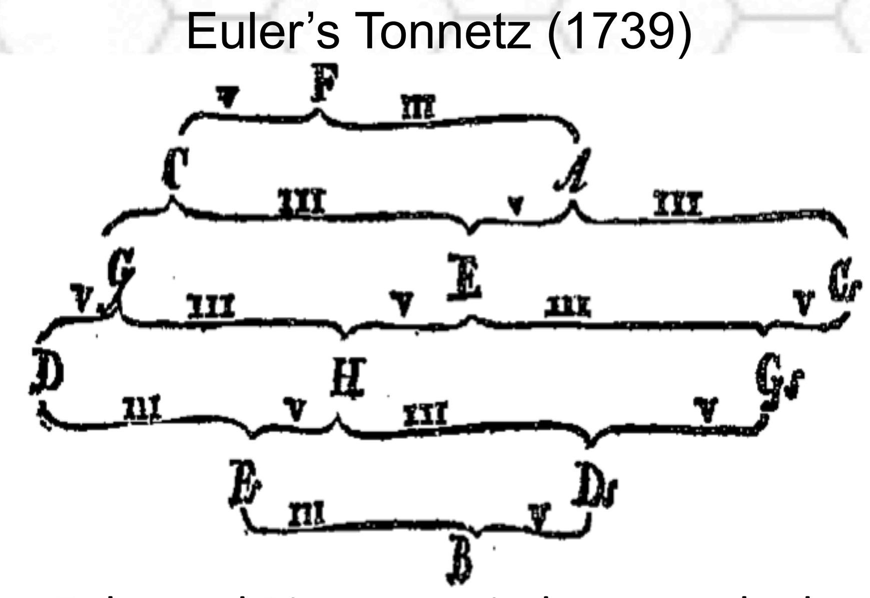
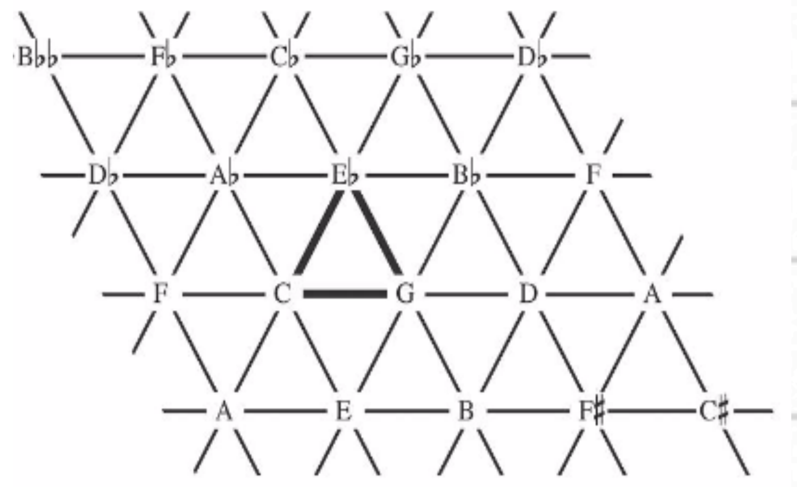
Hexagonally tiled Isomorphic Layouts and Chiral Vector

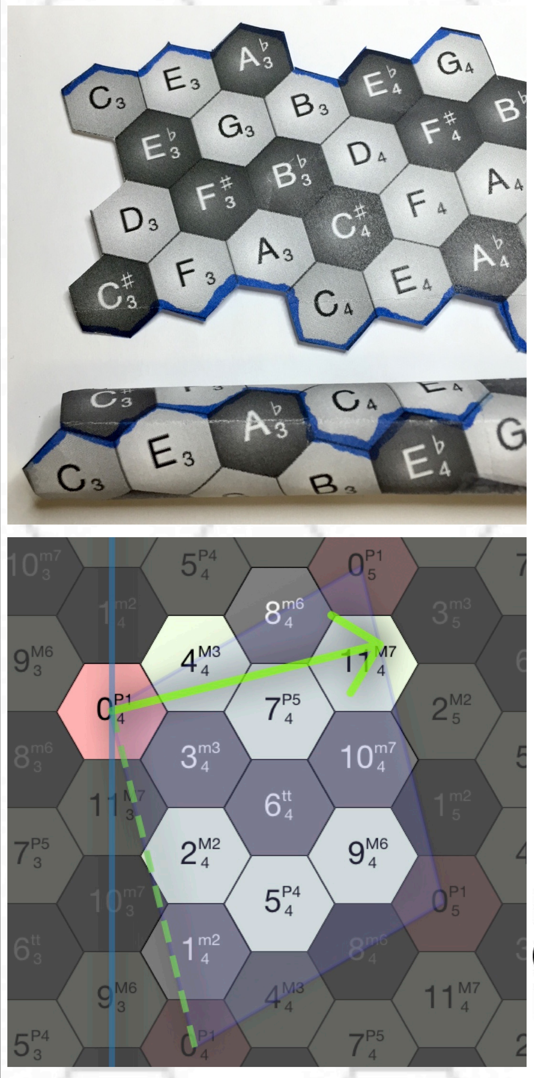
Euler's Tonnetz (1739)

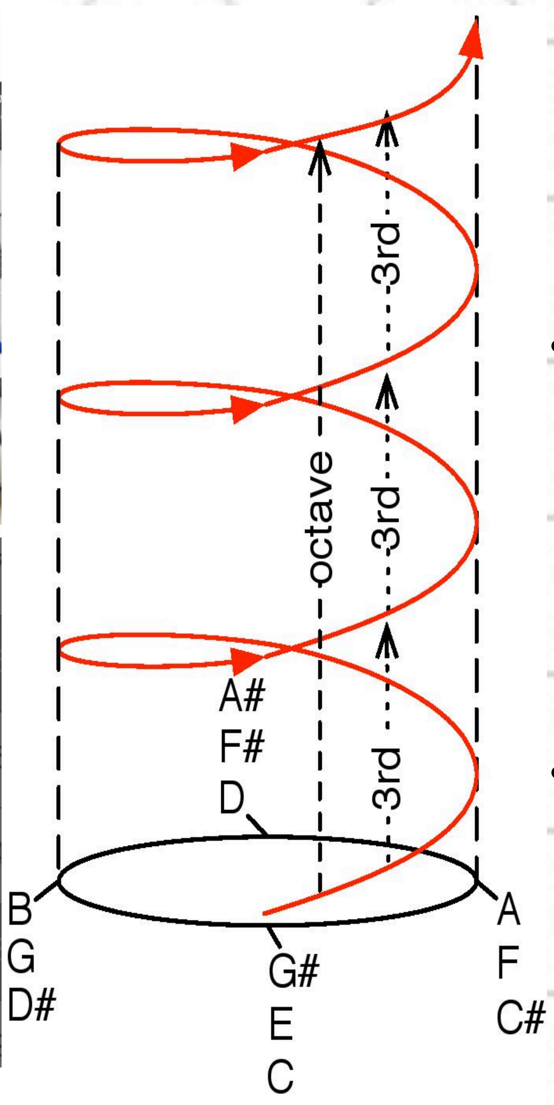


Riemann's Triangular lattice (1914)



- Euler and Riemann tried to map the harmonic arrangement of tones on lattices.
- According to the dual of Riemann's triangular lattice the hexagonal isomorphic layouts are generated to consistently present identical musical constructs such as chords regardless the beginning pitch.

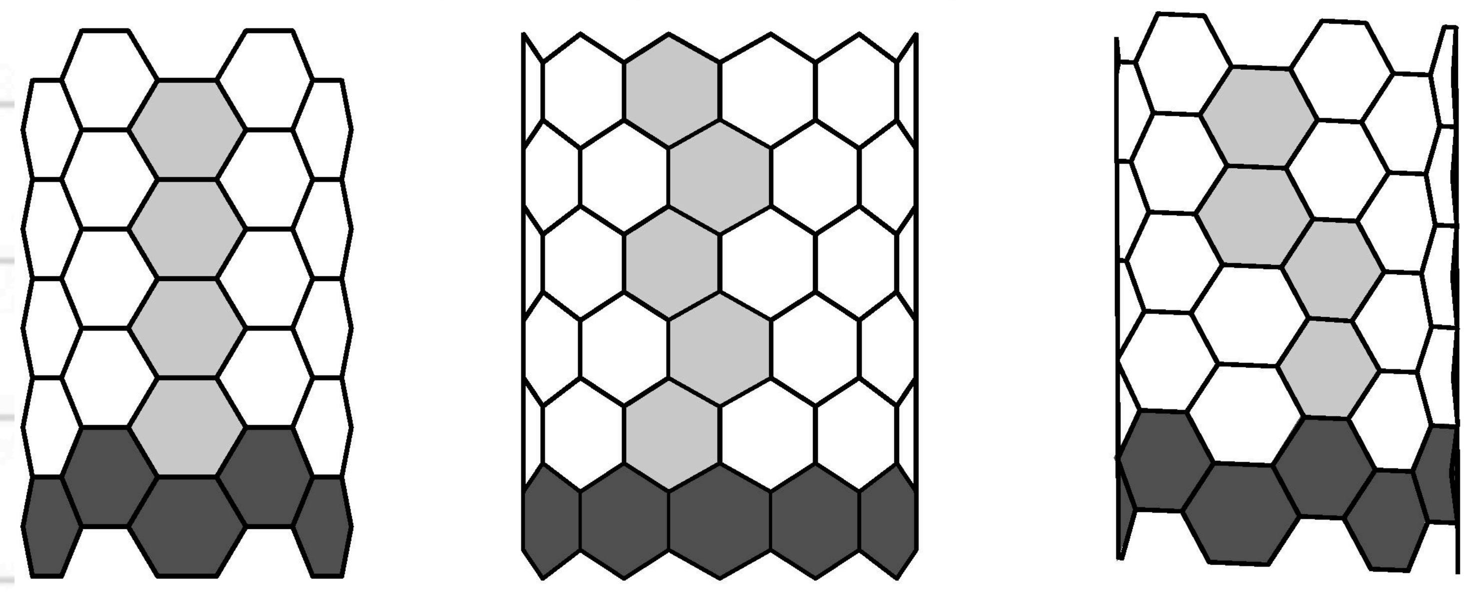




- By curling a planar hexagonal lattice which in specific direction, along the edges of hexagons, the resulting sheet becomes a cylinder.
- The vector that in isotone axis direction (a ray from one note of particular pitch passes through all same notes) is the chiral vector, which looks like spiral on the cylinder.
- Aim to map tone helixes we present cylindrical hexagonal lattices which have been extensively studied in context of carbon nanotube.

Cylindrical Hexagonal Lattices and Chiral Angle

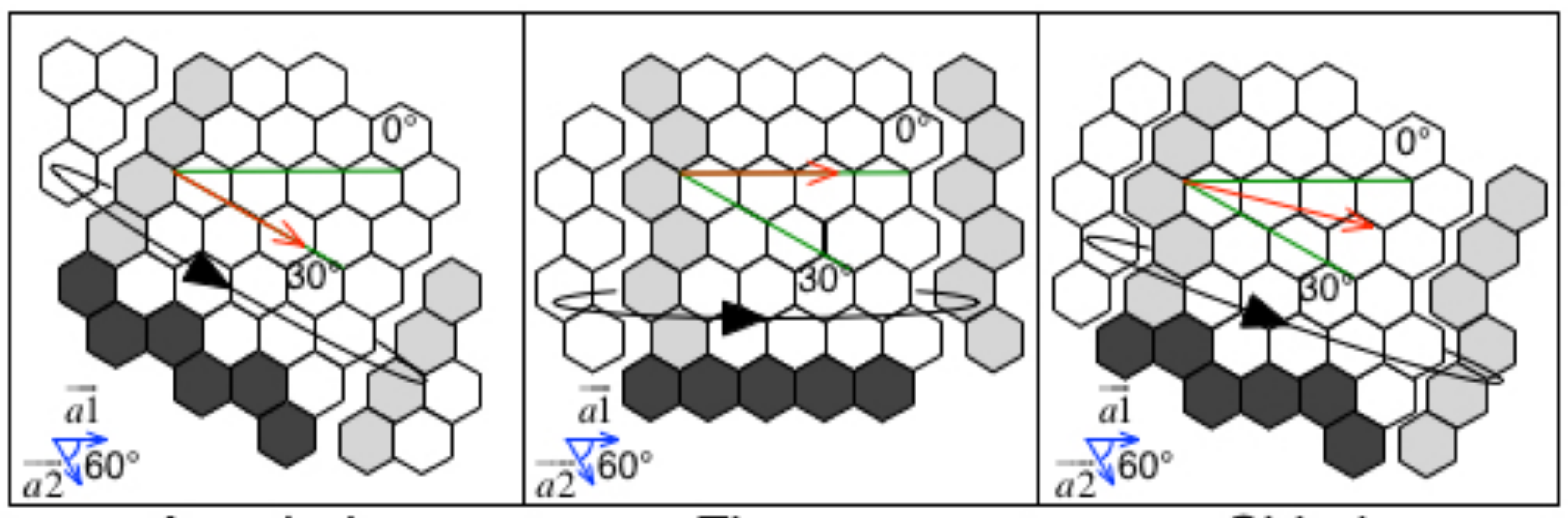
Three types of cylindrical hexagonal tubes



Armchair Zigzag Chiral

- A chiral tube (n,m), is defined by a chiral vector $\overrightarrow{Ch}$, indicating the orientation of the hexagonal lattice on the tube: $\overrightarrow{Ch} = n \bullet \overrightarrow{a1} + m \bullet \overrightarrow{a2}$, where $\overrightarrow{a1}$ and $\overrightarrow{a2}$ are two basis vectors separated by 60°.
- The lattices grouped by distinguishing chiral angle $\theta = \tan^{-1}\left[\frac{\sqrt{3}m}{m+2n}\right]$, as the angle between chiral vector and the Zigzag direction.
- Lattices: "Armchair", $\theta = 30^\circ$; "Zigzag", $\theta = 0^\circ$; Other chiral, $0^\circ < \theta < 30^\circ$

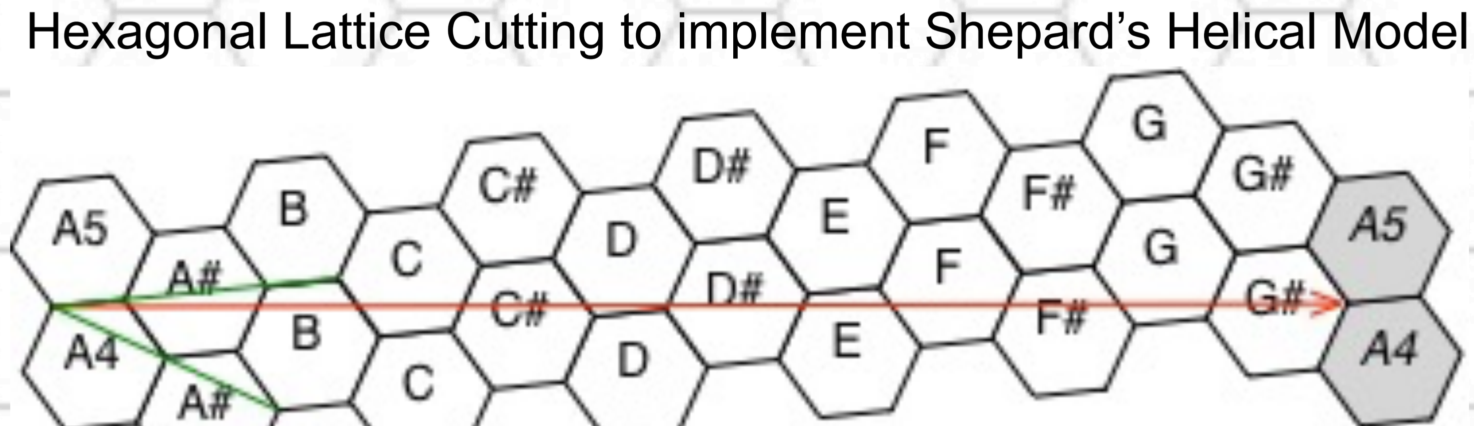
Three types of hexagon lattice cuttings



Armchair Zigzag Chiral

Implementation of Shepard's Helical Model

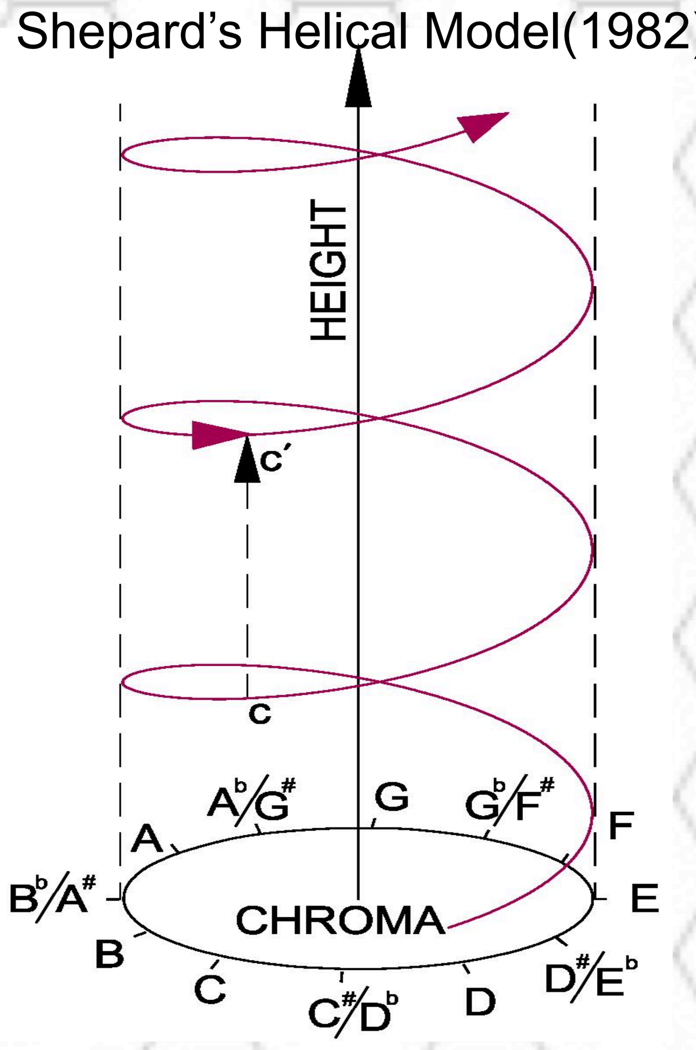
Hexagonal Lattice Cutting to implement Shepard's Helical Model



- According to Drobisch (1855)'s helix, Shepard introduced an equal-spaced helical model to arrange chromatic pitches over a regular, symmetrical, transformation-invariant surface and noted that this model is isomorphic.
- Considering in this model the pitch increases by semitones around the spiral, completing one turn of the spiral once per octave, the chiral angle can be

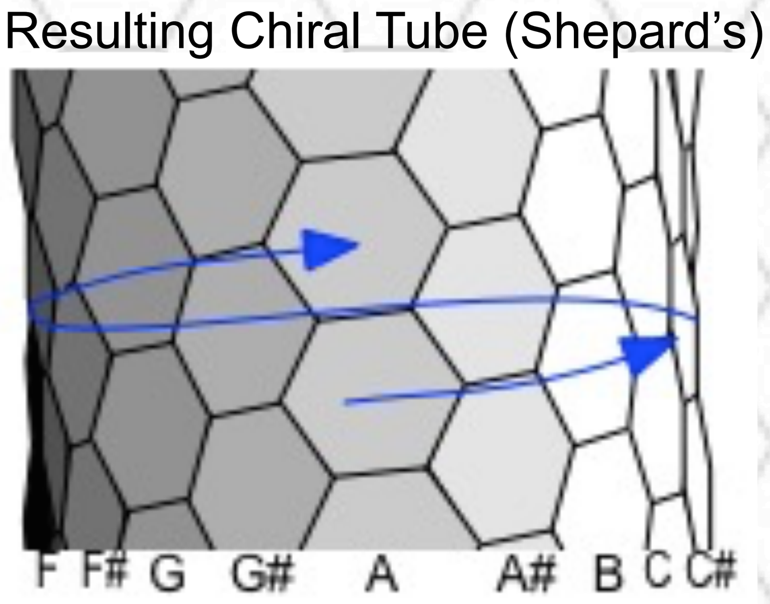
  calculated as $\theta = \tan^{-1}\left[\frac{\sqrt{3 \bullet 12}}{12+2}\right] = 23.2^\circ$
- The adjacent hexagons in one direction corresponding to semitones shown in the hexagonal lattice cutting.

Shepard's Helical Model(1982)



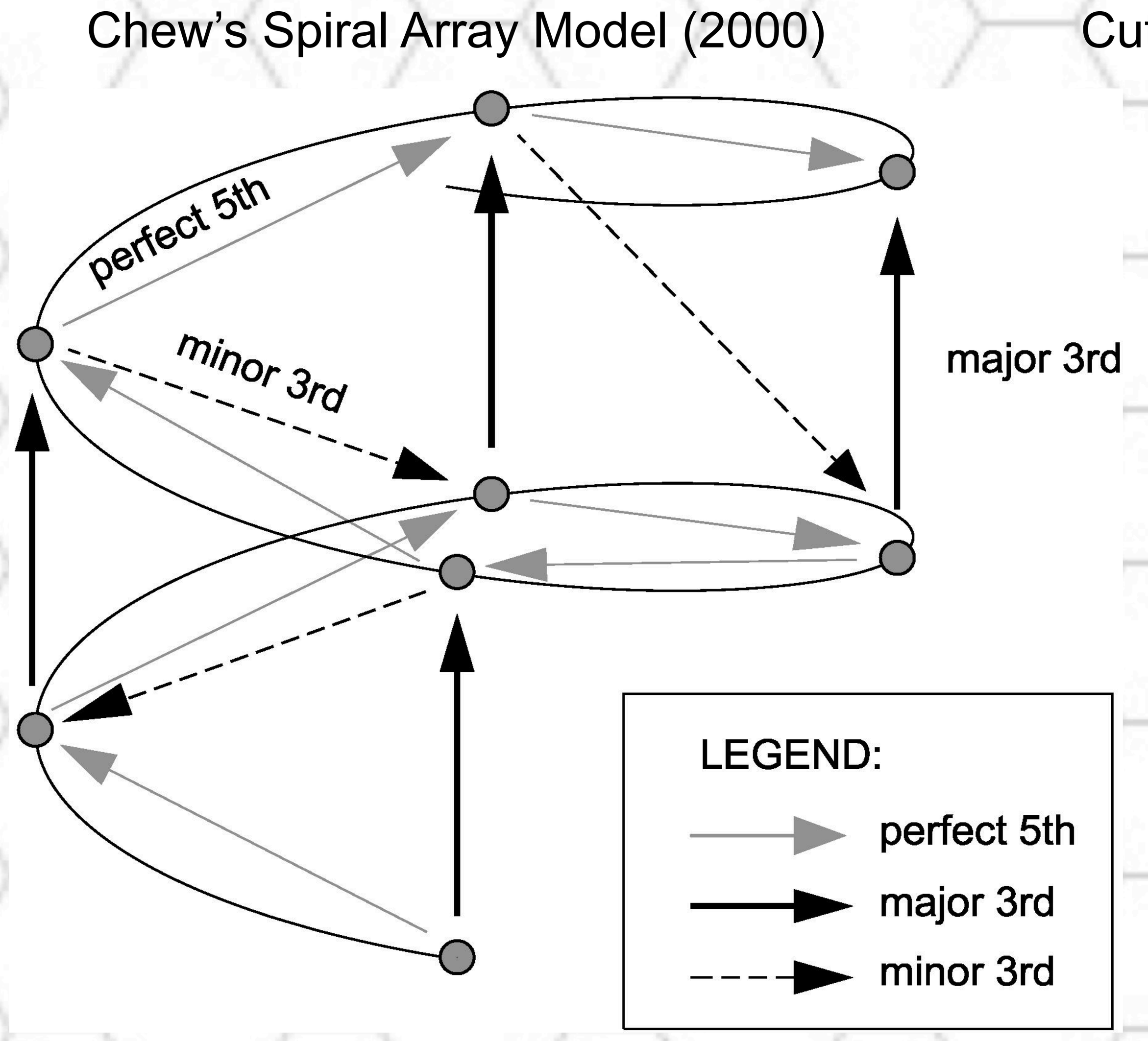
- Shepard's model allows a differential stretching or shirking of vertical extent of an octave of the helix relative to its diameter. So, we can accomplished this by allowing duplicates of the cutting, resulting in a larger-diameter tube.

Resulting Chiral Tube (Shepard's)

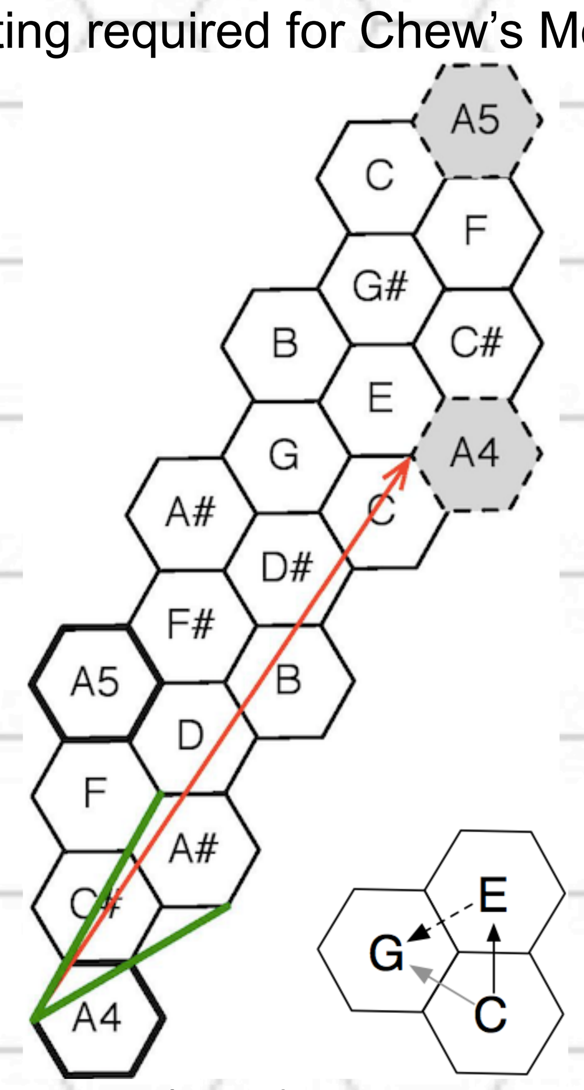


Implementation of Chew's Spiral Model and its Modification

Chew's Spiral Array Model (2000)

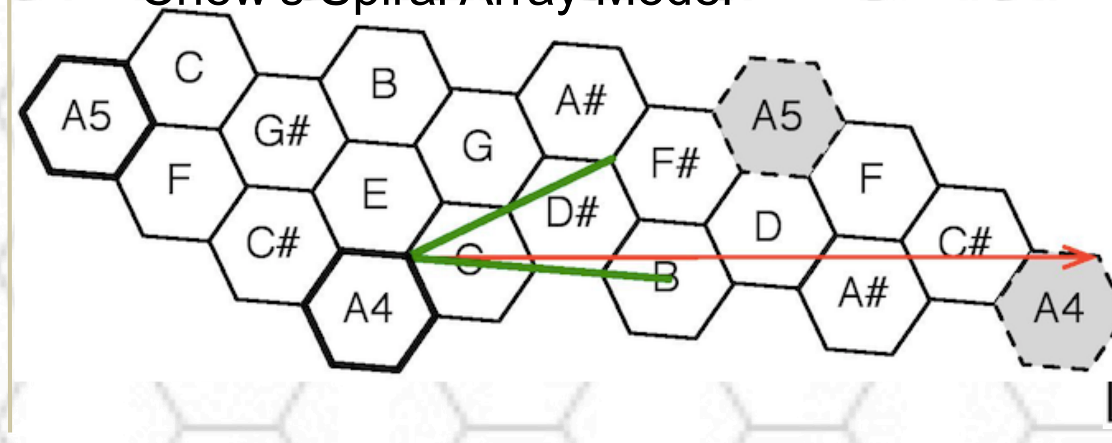


Cutting required for Chew's Model

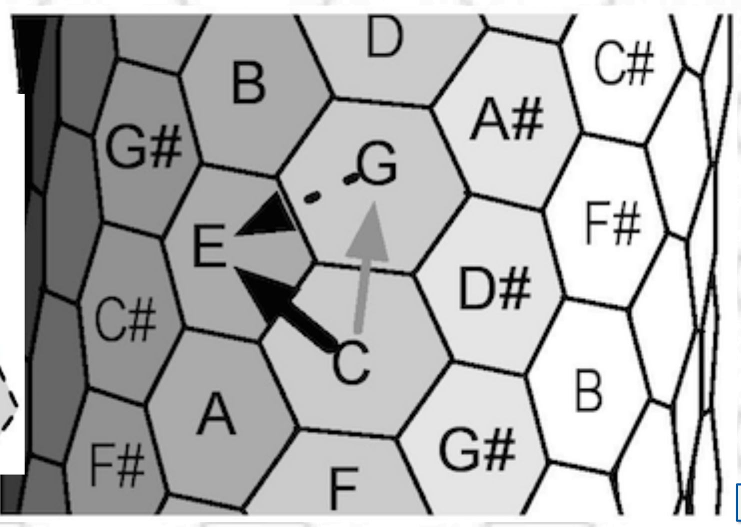


- Chew explored an abstract spiral model for mapping Tonnetz-based representations to the helix, providing an identical distance between each perfect fifth and another two between major / minor third by fixing angle with 90°.
- Since the chiral vector for this arrangement of the notes is not circumferential to the resulting tube, chew's model cannot be implemented by adjacent hexagons.
- If we make a modification by rotating and mirroring the model so that major thirds are along the spiral, and perfect fifths are in the vertical direction, it works properly.
- It may also possible to implement Chew's original pitch helix by allowing additional notes to appear between each note on the helix, and removing or ignoring the interspaced notes.

Cutting to implement modified Chew's Spiral Array Model



Resulting Chiral Tube (Modified Chew's)



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