On the Musical Opportunities of Cylindrical Hexagonal Lattices: Mapping Flat Isomorphisms Onto Nanotube Structures

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ABSTRACT

It is possible to position equal-tempered discrete notes on a flat hexagonal grid in such a way as to allow musical constructs (chords, intervals, melodies, etc.) to take on the same shape regardless of the tonic. This is known as a musical isomorphism, and it has been shown to have advantages in composition, performance, and learning. Considering the utility and interest of such layouts, an extension into 3D interactions was sought, focussing on cylindrical hexagonal lattices which have been extensively studied in the context of carbon nanotubes. In this paper, we explore the notation of this class of cylindrical hexagonal lattices and develop a process for mapping a flat hexagonal isomorphism onto such a lattice. This mapping references and draws upon previous explorations of the helical and cyclical nature of western musical harmony, but is not limited to 12-tone equal tempered scales.

1. INTRODUCTION

The tiling problem in music theory describes the challenge of using periodic or aperiodic congruent tiles to partition a plane into a representation of musical significance. One solution is to tile triads into vertices in an equilateral triangle lattice. Based on the dual map of the equilateral triangle lattices, congruent hexagonal lattices are introduced to present isomorphic layouts, which have the following characteristics of musical keyboard design [1]: Transposition Invariance, where each construct such as interval, chord and scale have the same geometric shape regardless of the root key; and Tuning Invariance, where all constructs must have the same geometric shape in all tunings of the continuum (which allows for an extension of this theory from common 12-tone equal tempered usage into microtonal tunings).

Since the tonnetz was first introduced by Euler in the 1700s, many physical instruments have been developed which use isomorphic layouts on flat hexagonal lattices, including the AXiS Keyboard, the Hex player and others. Most of the keyboards are not reconfigurable, only providing a single layout, as is the case with the Opal, the Thummer and the like. Keyboards like AXiS-49, AXiS-64 and Rainboard can be reconfigured to alternate isomorphisms, but they present only a subset of the layout in a fixed window, rather than the standard 88 keys / eight octaves, or more. The reason full sized isomorphic keyboards have not been developed is that the boundary shape of keys in one octave are not uniform. As discussed in Section 2, different isomorphic layouts have different base intervals, and therefore when the layouts are extended to include 8 octaves (or whatever constraint may be applied), the overall shape and structure will be different. In order to construct an instrument that has access to 8 octaves, in a reconfigurable arrangement, without shifting positions of notes, a very large number of controller buttons would be needed and the object itself would be unnecessarily expensive and unwieldy.

Taking into account the cyclic, helical, repetitive nature of musical harmony, especially as it appears on hexagonal musical isomorphisms, it is possible to represent all notes in a much tighter arrangement, by curling a flat hexagonal isomorphism through the third dimension and aligning repeating octaves along the circumference of the resulting cylinder. Drawing on mathematics already competed in the study of buckminsterfullerene (Carbon Nanotubes), this paper describes the mathematical theory behind the appropriate amounts of curl for different isomorphic layouts, and presents a framework for constructing any such cylindrical hex isomorphic layout.

The paper is organized as follows: First, we explore the current state of isomorphic keyboard layouts and present some historical examples. Next, we explore the mathematics of carbon nanotubes. Third, we explore the orientation of an isomorphism to a nanotube using pitch axes and chiral angles. We then present details of the various cases that arise with specific arrangements, classify those cases, and present solutions for each. Finally, we show some examples of nanotube isomorph curlings, and discuss some possible directions for instrument design and playablity.

2. ISOMORPHIC MUSICAL LAYOUTS

Isomorphic layouts are the product of research on the tiling problem, as well as geometries of musical theory. Euler [2] was the first to introduce such an arrangement, based on whole-number ratios of related frequencies mapped into a lattice, shown in Fig. 1. Later, Riemann presented a similar lattice [3], by using triangles to represent major and minor thirds, shown in Fig. 2. The dual of this triangular tessellation of vertices is a hexagonal grid.

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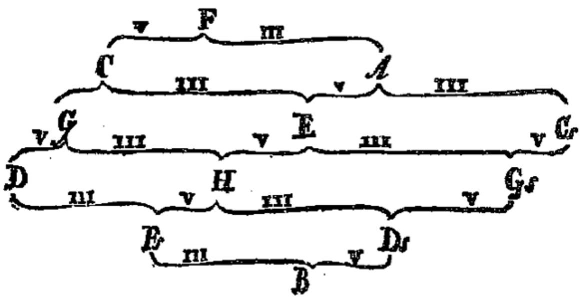


Figure 1: Euler's tonnetz.

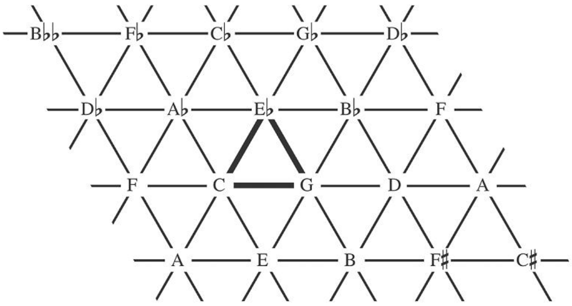


Figure 2: Riemann's triangular lattice.

Paul von Jankó designed a piano with horizontally whole tone and vertically semitone steps in 1882 [4], the arrangement of which is shown in Fig. 3. However, this piano did not achieve wide popularity because of the expense and weight of the instrument itself. The Wicki-Hayden layout was introduced as distinguishing seven white keys into two groups [5]. It benefits performance with shorten distance between keys in two groups, reducing learning time by unifying fingering into one pattern, and removing ambiguities by separating black keys into flat and sharp groups respectively. However, the keys in the Wicki-Hayden layout were not in a chromatic order, making it more difficult to learn for musicians used to adjacent semitones. Other popular isomorphic layouts such as Bajan, B-system, C-system, Gerhard and Park layouts are described in [6]. The AXiS keyboard, Opal, Thummer and Rainboard are physical constructions of these isomorphic layouts.

All of the isomorphic layouts mentioned above are based on flat, two-dimensional tessellations, which are usually hexagonal, but also can be rectangular, as in the case of the Jankó or Linnstrument layouts. Based on group theory and the tiling problem in mathematics, it is possible to map 2-d tessellations into higher dimensional space [7]. After carbon nanotubes (CNTs) were discovered from observations of formations of Fullerenes, the mathematical topology of carbon nanotubes became a subject of scrutiny in mathematical chemistry research [8, 9]. CNTs consisting of hexagons in their side-face are supersets of a 2-d hexagonal grid. By extensively exploring CNTs structure, overlaying existing hexagonal isomorphisms, and applying the constraints of transposition invariance and tuning invariance from isomorphism theory, we can construct musical keyboard layouts based around these shapes and perhaps open a new area of keyboard design.

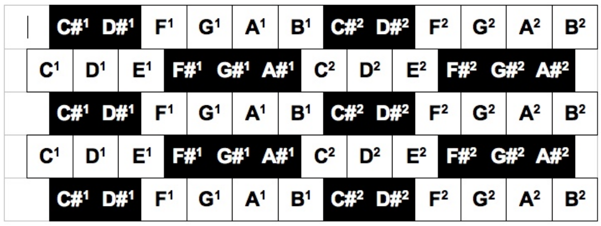


Figure 3: Jankó's piano

3. CYLINDRICAL HEXAGONAL TUBE LATTICES FROM A 2-D HEXAGONAL GRID

In this section we introduce two separate representations of coordinate systems for hexagonal lattices and see where they may meet. First, we introduce the notation used by carbon nanotube research, and second, we introduce the notation used by isomorphic musical keyboard research.

3.1 Hexagonal co-ordinates from nanotubes

If you start with a flat hexagonal lattice and begin curling, you will notice that there are a discrete number of ways that you can turn a sheet of hexagons into a tube of hexagons and have the hexagons line up properly. In order to make sure that the hexagons line up and make a complete cylindrical lattice, we explore the mathematics of the *chiral vector* a term taken from the study of nanotubes that indicates the direction in which hexagons will repeat.

A cylindrical hexagonal tube (n, m) , where $n \geq m$, is defined by a chiral vector. The definition of the chiral vector is

$$\overrightarrow{Ch} = n\overrightarrow{a1} + m\overrightarrow{a2}, \tag{1}$$

where $\overrightarrow{a1}$ and $\overrightarrow{a2}$ are two vectors within $60°$ on the grid.

In Fig. 4, we can see that $\overrightarrow{a1}$ and $\overrightarrow{a2}$ can be expressed in Cartesian coordinates (x, y) as

$$\overrightarrow{a1} = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) a \tag{2}$$

and

$$\overrightarrow{a2} = \left(\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) a, \tag{3}$$

where a is a the length between two vertices in a hexagon.

As shown in Fig. 4, each intersection point on a 2-d hexagonal grid can be represented by using these two vectors ($\overrightarrow{a1}$ and $\overrightarrow{a2}$). When we choose an origin, the other points are labelled with the hexagonal coordinate (n, m). In Fig. 4 these points are vertices of the lattice, but this vector representation is not limited to such vertices; it can be any point inside the hexagon or on the boundary. Section 4 describes how a vector chosen in such a representation may be used to describe the tube that is produced by cutting and curling the hexagonal lattice through the third dimension, and the three varieties of tube that can be generated depending in the way in which hexagons in the lattice line up and repeat.

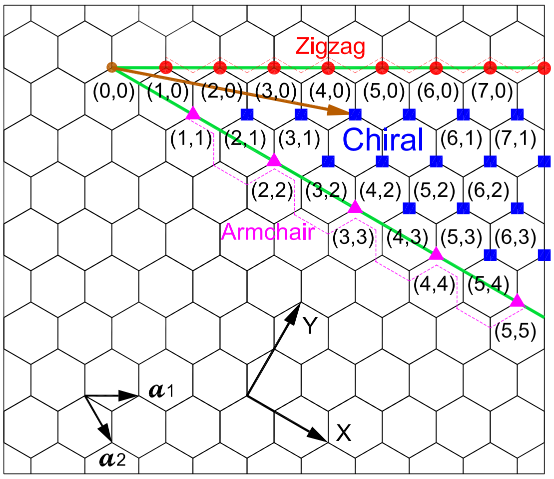


Figure 4: A 2-d hexagonal grid in Cartesian coordinates

3.2 Hexagonal co-ordinates from isomorphs

Musical isomorphisms have a different strategy for representation, one which is based on musical intervals. As described in Section 2, many different isomorphisms exist, depending on which musical interval is placed along which axis. In the original tonnetz, these intervals are major thirds and minor thirds, together making a major triad or minor triad depending on the order. Any two intervals can be combined to make an isomorphic layout, and a complete theory has been developed and presented in [10], wherein the LGD notation is introduced, with G being the greater of the two intervals, L being the lesser, and D being the difference. Hexagonal isomorphisms are thus represented based on their intervals as well as a possible rotation R and mirroring M factor, as well as shear S and an indication of the number of tones in the scale T, since this theory can be extended beyond the familiar 12-tone equal tempered scale into microtonal applications.

4. CHIRAL ANGLE AND THREE TYPES OF CYLINDRICAL HEXAGONAL TUBES

Hexagonal lattices can be curled into tubes in three different ways, shown in Fig. 5, based on the angle that we choose in Fig. 4. If we choose hexagons that are flat against each other and wrap them around to form the circumference of the tube, the pointed ends of the hexagons stick out and we call this "zigzag". If we choose to go in the other direction, with the pointed ends of the hexagons touching, we get a notched tooth appearance for the end of the tube, and this is called "armchair". If, instead, we spiral the hexagons around in a helix so that one layer builds upon the next, these are other chiral tubes, and there are many different ways we could do this.

We can also group into these three types by distinguishing the chiral angle Θ, as the angle between the chiral vector and the zigzag direction shown in Fig. 4:

$$\Theta = \tan^{-1}\left[\frac{\sqrt{3}m}{m+2n}\right] \tag{4}$$

By following Equation (4), the three types are:

Armchair $(m = n)$: $\Theta = \tan^{-1}\left[\frac{1}{\sqrt{3}}\right] = 30°$, the trace shown by purple dash line with purple triangles in Fig. 4.

Zigzag $(m = 0)$: $\Theta = \tan^{-1}[0] = 0°$, the trace shown by red dash line with red dots in Fig. 4.

Other chiral tubes (called "Chiral"): $0° < \Theta < 30°$, the area between the zigzag and armchair angles shown in Fig. 4.

Once cut and curled, these three vectors produce three types of cylindrical hexagonal tubes, shown in side view in Fig. 5.

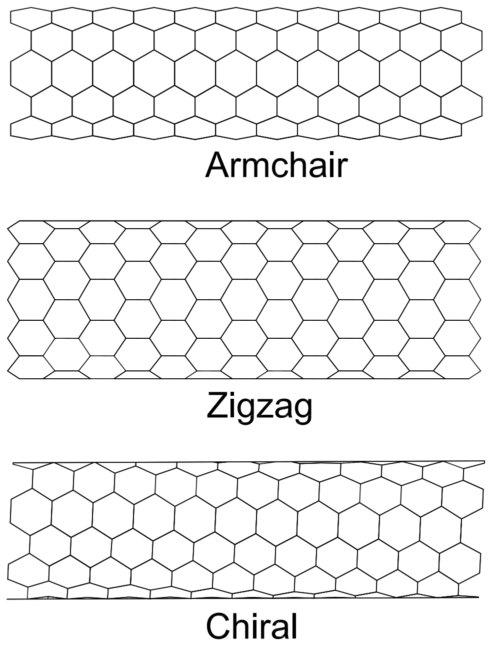


Figure 5: Three types of cylindrical hexagonal tubes

5. TUBE LENGTH AND DIAMETER

Because we need hexagons to line up perfectly in order to produce a viable tube lattice, we can't produce a tube of just any size. For example, if we consider the zigzag tube, we can only produce tubes which are a whole number of hexagons "around". The same holds true for any type.

The diameter of a tube is decided by the length of chiral vector. From equations (1), (2) and (3), the length of chiral vector is the peripheral length of the tube:

$$\|\overrightarrow{Ch}\| = \sqrt{3}a\sqrt{n^2 + nm + m^2}, \tag{5}$$

where a is the length of an edge between two vertices in a hexagon. The diameter of such a tube is therefore:

$$D = \frac{\|\overrightarrow{Ch}\|}{\pi} = \frac{\sqrt{3}a\sqrt{n^2 + nm + m^2}}{\pi}, \tag{6}$$

and for the Armchair $(m = n)$ and Zigzag $(m = 0)$ cases, Table 1 shows the corresponding tube parameters.

Tube	Chiral Length	Tube Diameter
Armchair	$3na$	$3na/\pi$
Zigzag	$\sqrt{3}na$	$\sqrt{3}na/\pi$

Table 1: Parametric descriptors of chiral vector length and tube diameter for armchair and zigzag cases.

6. MAPPING A 2-D HEXAGONAL GRID INTO A 3-D CYLINDRICAL HEXAGONAL LATTICE

If we are to be successful in curling a flat isomorphism into a tube, we must ensure that the intervals are preserved. For example, if we use a harmonic table layout (a well known layout closely resembling the original tonnetz), then adjacent hexagons must be minor thirds, major thirds or fifths. If we were to wrap this into a tube, then if we start at a given note and proceed around the tube, we must end up back at the note we started with. This puts a strict constraint on the way that isomorphisms can be wrapped: *The circumference of the tube must be in a direction in which repeated notes will be found on the original flat layout*.

Conveniently, the *LGD* representation of isomorphisms provides such a direction. In [10], the isotone axis is defined as a line contains all the instances of a particular note in an isomorphic layout. The pitch axis is a line perpendicular to the isotone axis, and is the direction in which pitch increases. Figures 6 through 13 show examples of some of the more common hexagonal isomorphisms, with their pitch axis indicated with a green arrow, and their isotone axis indicated with a dashed green line.

By setting the chiral vector direction in hexagonal coordinates (n, m), to be equal to the isotone axis in the *LGD* representation of a musical isomorphism, with an appropriately chosen chiral vector length, each isomorphic layout can be mapped from a 2-d grid into a 3-d cylindrical hexagonal lattice.

6.1 Mapping the isotone axis to the chiral vector

In *LGD* notation, either the isotone axis range or pitch axis range can be transposed by using rotation and reflection. However, since the hexagon is a member of the dihedral group [1], it is possible to focus on the area in hexagonal coordinates (n, m) with Θ (the chiral angle) as $0° \leq \Theta \leq 30°$. Besides, either *D*, $-L$ directions or $-D$, *L* directions has 60 degree opening which is the same as $\overrightarrow{a1}$, $\overrightarrow{a2}$ vectors. We can therefore set the isotone axis in each isomorphic layout equal to a chiral vector direction, by mapping *D* and -*L* into $\overrightarrow{a1}$ and $\overrightarrow{a2}$ directions respectively.

We can now define a new notation (D, L) which fully represents the isomorphic cylinder corresponding to the isotone axis range in the *LGD* notation. Correspondingly, the vector perpendicular to the chiral vector which is called the *translation vector* goes in the same direction as the pitch axis, and represents the direction of the axis of the resulting cylinder.

[1] Dihedral group: A mathematicaly defined set of symmetries of a regular polyhedron which includes reflection and rotation

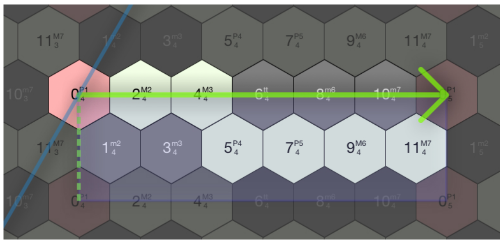


Figure 6: Jankó (1,1)

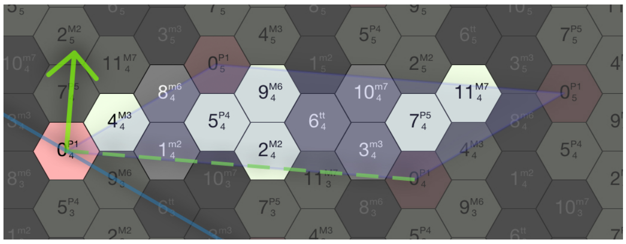


Figure 7: Harmonic Table (4,3)

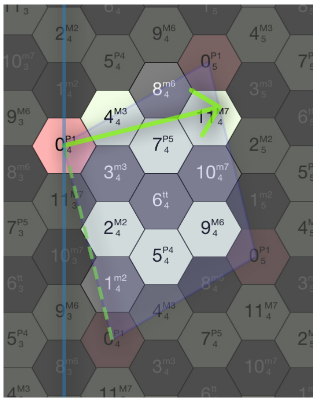


Figure 8: Gerhard (3,1)

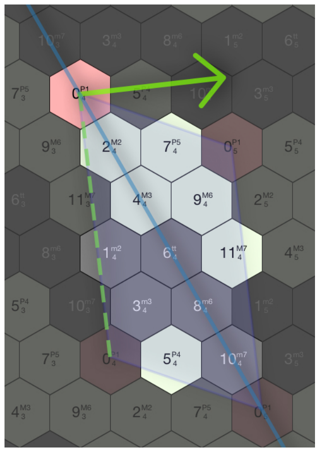


Figure 9: Park (3,2)

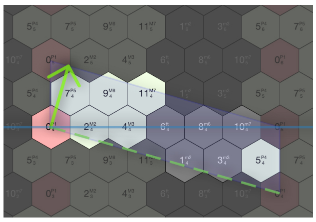


Figure 10: Wicki-Hayden (5,2)

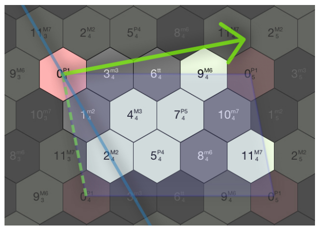


Figure 11: Bajan (2,1)

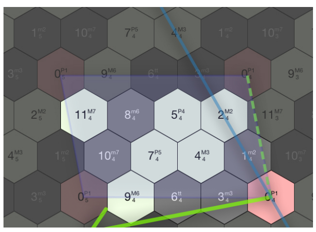


Figure 12: B-system (2,1)

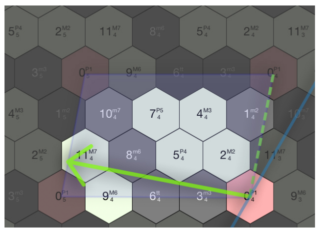


Figure 13: C-system (2,1)

We can consider a subset of an isomorphic layout consisting of a single copy of each note from a single octave (in this case 12 notes since the system we are using is 12-tone equal tempered, but this could be extended to microtonal systems). This sample patch represents the smallest unit that can be considered when curling such an isomorph into a tube. Along the isotone axis, these patches repeat identically, and represent a further constraint - each tube must have around its circumference a whole number of copies of this patch.

Considering Figs. 6–13 again, we also see a blue straight line. This line represents what would be the zigzag chiral direction, and so the angle between this and the dashed green line represents the chiral vector. We have seen already that the dashed green line represents the isotone axis of the layout, and so we can see that each layout also maps to a cylindrical hexagonal lattice structure with a specific chiral vector.

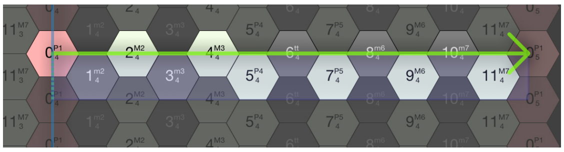
The specific chiral angles of these common isomorphic layouts (in degrees to two significant digits) is calculated using equation (4), and are shown in Table 2.

Layout	(D, L)	Chiral angle
Jankó	(1,1)	30.00°
Harmonic Table	(4,3)	25.29°
Gerhard	(3,1)	13.90°
Park	(3,2)	23.41°
Wicki-Hayden	(5,2)	16.10°
Bajan	(2,1)	19.10°
B-system	(2,1)	19.10°
C-system	(2,1)	19.10°

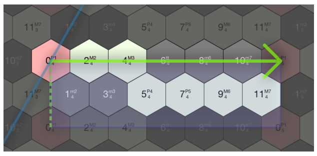
Table 2: The chiral vector of typical isomorphic layout.

6.2 Edge cases: Zigzag and Armchair

There are two special cases mentioned in [10]. The first one is where L= 0, which only happens for intervals 0,1,1 in *LGD* notation. This case results in a **Zigzag** type lattice (1,0). The second case is where D=L, which happens for intervals of 1,2,1 in *LGD* notation. This case makes the **Armchair** type lattice (1,1). The samples of those two special cases are shown in Fig. 14, and both can be seen to be instances somewhat similar to a Jankó layout.



(a) **Zigzag** (1,0)



(b) **Armchair** (1,1)

Figure 14: Two special cases exist in the lattices

7. BENEFITS OF A 3-D CYLINDRICAL HEXAGONAL LATTICE

We now have two new parameters that can be used to describe isomorphisms that have been curled into cylinders: The chiral angle, and the diameter. Between these two, it will be possible to explore the musical and ergonomic characteristics of different cylindrical isomorphisms, construct them into physical instruments, give them to musicians to play with, and characterize them based on playability, learnability, and expression. This exploration will be undertaken in future work, but we can begin with a theoretical discussion of some of the different playing modes and characteristics.

Presented here are three potential benefits of using a cylindrical hexagonal isomorphic lattice.

7.1 Boundary conditions and note reachability

One of the primary features of any arrangement of note actuators on a musical instrument is to make notes reachable. Adding additional manuals to an organ or additional strings to a bass guitar, for example, serve two purposes: to extend the range of the instrument, but also to make more notes available with less hand travel. On a traditional piano keyboard, only a little more than an octave of notes is available in any one hand position, and the ability to accurately move your hand to a new position is a critical stage in studying the piano.

Isomorphic layouts have the potential to be more compact than existing instruments, making more notes available in a single hand position and making all notes a smaller distance from the centre of the layout. However, any attempt to construct a reconfigurable hexagonal instrument that can present different isomorphisms comes to a challenge: each isomorphism potentially has a different *boundary*, that is, the overall shape of the entire layout showing all notes. Figure 15 shows the boundaries of two similar layouts.

It would be difficult to create a reconfigurable musical instrument that could represent both of these layouts to their top and bottom boundaries for two reasons. First, the angle of the boundaries is different; and second, the orientation of the hexagons is different: Wicki-hayden uses a “horizontal” layout where adjacent hexagons share a vertical face, while the Harmonic Table layout uses a “vertical” layout. Indeed, both layouts represent infinite duplications of notes to the left and right, at different angles, which would add to the challenge of manufacturing such an instrument.

Considering the parallelograms shown in Figs. 6 through 13, and extending these by repeating along the isotone axis and extending along the pitch axis, we see that each of the popular layouts will have a very different boundary shape. These boundary shapes are compared in Fig. 16. This is also related to the *shear*, a characteristic of an isomorphic layout, described in [11].

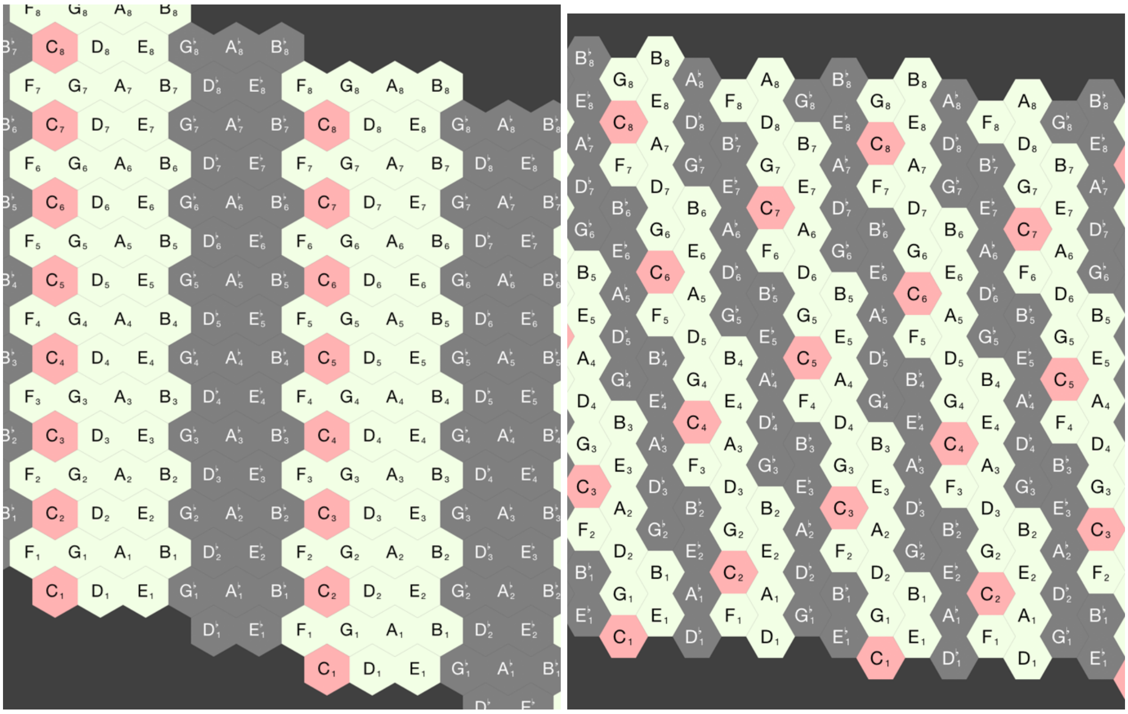


Figure 15: 8-octave boundaries of two popular isomorphisms. Left: Wicki-hayden. Right: Harmonic Table

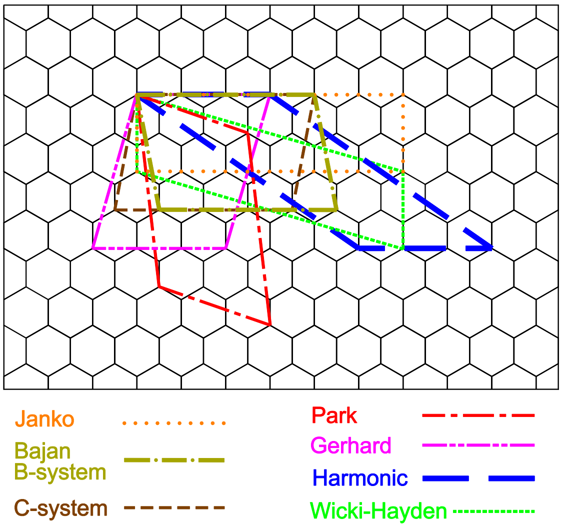


Figure 16: Parallelograms of typical isomorphic layouts

Layout	Diameter
Jankó	$\frac{3a}{\pi}$
Harmonic Table	$\frac{\sqrt{111}a}{\pi}$
Gerhard	$\frac{\sqrt{39}a}{\pi}$
Park	$\frac{\sqrt{57}a}{\pi}$
Wicki-Hayden	$\frac{\sqrt{117}a}{\pi}$
Bajan	$\frac{\sqrt{27}a}{\pi}$
B-system	$\frac{\sqrt{27}a}{\pi}$
C-system	$\frac{\sqrt{27}a}{\pi}$

Table 3: Diameters of eight typical isomorphic layouts, where a is the length of one side of a hexagon.

7.2 Wrapping infinite repetitions into tubes

Considering again the boundary shape of each isomorphism, it should be clear that the previous discussion on nanotube mapping and chiral angle can be simplified by considering an infinite sheet of repetitions of notes, and rolling that sheet in such a way that the repetitions coincide around the circumference of a tube. It should also be clear that the diameter of these tubes will be constrained to a whole number multiple of the distance between identical notes in the same octave. Table 3 shows the diameter of the tube corresponding to each of the layouts under discussion, calculated using equation (6).

When considering the construction of a physical instrument, given that there are different tube sizes required, there are two options: allow the size of the instrument to change; or allow the size of the buttons to change. Both present significant technical challenges that are not addressed in this paper and left for future work.

To map a specific isomorphic layout onto a tube with a given diameter, the length of the side of the hexagonal tiles (a) must be changed. As an example, consider the situation where two different isomorphisms are to be mapped onto a tube of a given size. The ratio of size of two hexagonal buttons can then be calculated from Table 3. Mapping *Gerhard* and *Wicki-Hayden* on the same tube, we must set:

$$\frac{\sqrt{39}a1}{\pi} = \frac{\sqrt{117}a2}{\pi}$$

Which also assumes that the both cylinders are using the same number of copies of the base set of notes around the

Figure 17: Tube size varied by the number of duplicates. (from left) 4 copies, 3, 2, and 1.

circumference. Simplifying, we get:

$$\frac{a1}{a2} = \frac{\sqrt{3}}{1}$$

which means the size of buttons in *Gerhard* layout is $\sqrt{3}$ times bigger than that in *Wicki-Hayden* layout, given the same tube diameter.

7.3 Tube Size and Note Duplication

As already discussed, the size of the tube for any given isomorphism will depend on the number of copies of the base parallelogram that are included around the circumference of the tube. This choice is aesthetic and can be used to influence playability, interaction, note availability, button size, and other factors.

Figure 17 presents a set of possible tubes from the same isomorphic layout, in this case the Gerhard layout. The only difference between tubes is the number of duplicates that go around the circumference of the tube. If a single copy is used, the tube is quite narrow and each note appears exactly once on the entire structure. Adding more duplicates makes the tube larger, but does not change the shape of any harmonic constructs on the tube.

8. PLAYABILITY MODES

Fingering on a curved keyboard can be a solution for some particular isomorphic layouts which were considered having "fingering difficulties" on 2-d planar keyboard, but this will require further study to conclusively prove. One can imagine a controller constructed with the ability to "roll" across a table or surface (Fig. 18), allowing different notes to become available at different times. With the appropriate layout, this could be an additional compositional or performance function, modulating key or tonality or adjusting other musical parameters.

It is also possible to imagine a larger cylinder with keys tiled on the inside of the surface instead of the outside. This could produce a compelling stage presence with players performing inside the lattice, and playing on the inner surface. The inside and outside tiling are shown in Fig. 19.

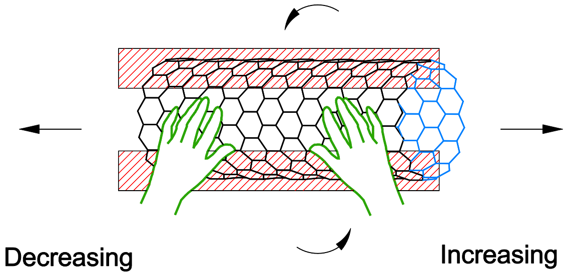


Figure 18: An appropriate area along either decreasing or increasing octave direction

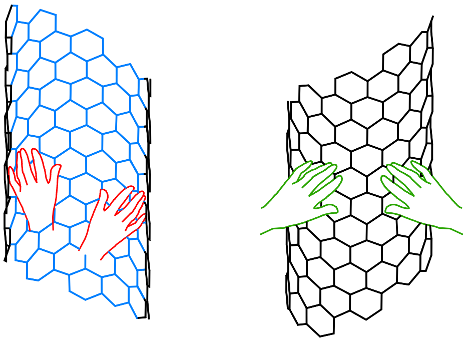


Figure 19: Playing on inner (left) or outer surface

9. CONCLUSIONS AND FUTURE WORK

In this paper we have presented a discussion on turning the cyclic nature of tonnetz-style isomorphic discrete note layouts into a true cylindrical cycle, by setting the isotone axis of an isomorphism equal to the chiral vector direction of a nanotube. By choosing the intervals on the isomorphic axes, and by changing the number of duplicates and the size of buttons on each cylindrical hexagonal lattice, it is possible to create a wide variety of tubes of different sizes and structures, each of which maintains the strong constraints of isomorphic note arrangements while offering the possibility of new playing interfaces, compositional structures, and learning tools.

Future work on this topic will begin with brute-force generating a set of tubes for all possible isomorphisms, based on the completeness work in [10]. With this, we can explore the similarities and differences between tube layouts, as well as the musicality, playability, and interaction modes of these tubes. Next, we plan to choose some of the tubes with the greatest potential for new modes of interaction and physically construct new controllers based on this theory, and provide these to musicians, composers, and students, to explore and study. We will also formally study the musical and educational benefits of these tube structures. A long-term goal is to explore the possibility of creating a single reconfigurable tube for which the diameter and chiral angle can be modified in real time.

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