Formal Concept Analysis based on Hierarchical Class Analysis

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Abstract

The study of concept formation and learning is a central topic in cognitive informatics. Formal concept analysis can be viewed as an approach on this topic based on a formal context. In this paper, we study some basic data units for formal concept analysis based on hierarchical class analysis. In a formal context, objects and attributes can be grouped into equivalence classes based on equivalence relations. The order relations define the hierarchical structures of the equivalence classes. There exist some basic sets of objects and attributes that are used to not only reflect the equivalence relation and order relation, but also constitute all formal concepts derived from the formal context. The properties of these basic sets are studied and analyzed.

1 Introduction

Today, information is not only concerned as a probabilistic measure of the quantity of messages or signals through a transmission channel, but also regarded as data and knowledge representation, storage, and processing [13]. Traditional approaches on informatics are no longer sufficient. Kinsner argued that entropy measures for informatics were not suitable enough for cognition [9]. Structural and contextual metrics as well as entropy-based metrics should be considered.

Currently, the studies of informatics have attracted much attention from many different research areas, such as psychology, artificial intelligence, linguistics, neuroscience, and so on. As a multidisciplinary research area, cognitive informatics deals with information and intelligence and related processing mechanisms in humans and computers via interdisciplinary approaches [13].

Many studies aim at establishing theoretical foundations of cognitive informatics by simulating human knowing and thinking [10, 14, 17]. For example, a layered reference model of the brain (LRMB) developed by Wang *et al.* explains the mental mechanisms and cognitive processes of the natural intelligence [15].

Concepts may be viewed as the basic units of thoughts in human intelligence. The theoretical study and analysis of concept, concept formation and concept learning is central to cognitive informatics [10, 14, 17].

Many approaches on concept formation, learning and analysis are proposed. Recently, formal concept analysis are paid more attention. Formal concept analysis is introduced by Wille in the early 80's [16]. It is developed based on a formal context. It is not only a method for data analysis and knowledge representation, but also a formal formulation for concept formation and learning [7, 16, 18, 20]. It provides a formal and graphical way to describe, characterize and summarize data as concepts. In formal concept analysis, a concept is a pair of a set of objects and a set of attributes that uniquely associate, determine, and describe each other.

In this paper, we study some basic sets of objects and attributes in a formal context for formal concept analysis based on hierarchical class analysis. The hierarchical class analysis is introduced by using formal set-theoretical formulations. The basic sets of objects and attributes are analyzed with respect to formal concept analysis.

Hierarchical class analysis is an approach on analysis of data in a formal context [2, 3, 4, 6]. It represents the hierarchical structure of the data in a numerical and a graphical way based on a set-theoretical framework. A binary relation in a formal context associates objects with attributes. The equivalence relation among objects or attributes can be defined based on the binary relation. The equivalent objects and equivalent attributes are grouped into classes of objects and attributes, respectively. Furthermore, an order relation also can be defined based on the binary relation. The hierarchies of the object and attribute classes are constructed based on the order relation.

By analyzing the definition of the equivalence classes, we find that there exist some basic sets of objects and attributes, which reflect the equivalence relation and order relation on objects and attributes, and can be used to form all the concepts induced from the formal context. This study links hierarchical class analysis and formal concept analysis together, and provides a in-depth understanding of formal concept analysis. Moreover, our studies can be considered as providing some new tools for concept formation, analysis and learning.

2 Formal Contexts

An information table consists of a nonempty and finite set of objects, a nonempty and finite set of attributes, and the values of objects on attributes. A binary information table is an information table in which each entry contains value 0 or 1, such as the example illustrated in Table 1. Every non-binary information table can be represented by a binary information table [1].

	a	b	c	d	e	f	g	h	i	j	k	l
1	1	1	0	0	1	1	0	0	0	0	0	1
2	1	1	0	0	1	1	0	0	0	0	0	1
3	1	1	0	0	1	1	0	0	0	0	0	1
4	0	0	1	1	1	1	0	0	0	1	1	1
5	0	0	1	1	1	1	0	0	0	1	1	1
6	0	0	0	0	0	0	1	1	1	1	1	1
7	0	0	0	0	0	0	1	1	1	1	1	1
8	0	0	1	1	1	1	1	1	1	1	1	1
9	0	0	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1	1

Table 1. A binary information table taken from [4]

Let U and V be two finite and nonempty sets. U is called the universe of objects and represents the set of all objects. V is called the universe of attributes and represents the set of all attributes. A binary relation R between the universes U and V is a subset of the Cartesian product $U \times V$. For an object $x \in U$ and an attribute $y \in V$, if $(x, y) \in R$, written as xRy, we say that the object x possesses the attribute y, or the attribute y is possessed by the object x. Based on the binary relation R, the objects and attributes can be described by each other. The triplet $S = (U, V, R)$, representing a binary information table, is called a formal context [7].

In a formal context, given an object x, the set of attributes that are possessed by an object x is denoted by xR:

$$xR = \{y \in V \mid xRy\} \subseteq V.$$

The notation of xR presents an association between an object x and a particular attribute set xR in which each attribute is possessed by the object x.

For an attribute y, the set of objects that possess y is denoted by Ry:

$$Ry = \{x \in U \mid xRy\} \subseteq U.$$

Ry demonstrates an association between an attribute y and an object set Ry.

By extending the notations xR and Ry, we can establish relationships between sets of objects and sets of attributes [7, 16]. This leads to two operators, one from 2^U to 2^V and the other from 2^V to 2^U.

Definition 1 *Suppose (U, V, R) is a formal context. For a set of objects X, we associate it with a set of attributes:*

$$\begin{aligned} X^* &= \{y \in V \mid \forall x \in U (x \in X \Longrightarrow xRy)\} \\ &= \{y \in V \mid X \subseteq Ry\} \\ &= \bigcap_{x \in X} xR. \end{aligned} \qquad (1)$$

For a set of attributes Y, we associate it with a set of objects:

$$\begin{aligned} Y^* &= \{x \in U \mid \forall y \in V (y \in Y \Longrightarrow xRy)\} \\ &= \{x \in U \mid Y \subseteq xR\} \\ &= \bigcap_{y \in Y} Ry. \end{aligned} \qquad (2)$$

For simplicity, the same symbol * is used for both operators. The actual role of the operators can be easily seen from the context.

Düntsch and Gediga referred to * as a modal-style operator, called the *sufficiency operator* [5, 8]. In fact, it reflects a unique association between objects and attributes. Moreover, X^* for a set of objects X is the maximal set of attributes possessed by all objects in the set X. That is, the set X^* consists of necessary attributes of an object in X. In the other words, an object in X must have attributes in X^*. Similarly, Y^* for a set of attributes Y is the maximal set of objects that have all attributes in Y. That is, an attribute in Y must have objects in Y^*, and Y^* consists of necessary objects of an attribute in Y.

More studies of sufficiency operator and its relationships with other modal-style operators based on a formal context are provided by Yao [19].

The sufficiency operator has the following properties [7]: For $X, X_1, X_2 \subseteq U$ and $Y, Y_1, Y_2 \subseteq V$,

$$\begin{aligned}
(1) \qquad & X_1 \subseteq X_2 \Longrightarrow X_1^* \supseteq X_2^*, \\
& Y_1 \subseteq Y_2 \Longrightarrow Y_1^* \supseteq Y_2^*, \\
(2) \qquad & X \subseteq X^{**}, \\
& Y \subseteq Y^{**}, \\
(3) \qquad & X^{***} = X^*, \\
& Y^{***} = Y^*, \\
(4) \qquad & (X_1 \cup X_2)^* = X_1^* \cap X_2^*, \\
& (Y_1 \cup Y_2)^* = Y_1^* \cap Y_2^*,
\end{aligned}$$

By Property (1), it is easy to know that if a set of objects is larger, then its associated attribute set is smaller. Similarly, if a set of attributes is larger, then its associated object set is smaller.

3 Formal Concepts Analysis

In the classical view, a concept is defined by a pair of extension and intension. In a formal context, a set of objects is referred to as extension and represents instances of a concept. A set of attributes is referred to as intension and characterizes the features or properties of a concept.

Formal concept analysis focuses on a unique association between the extension and intension [7, 16]. That is, an object set and an attribute set can uniquely associate with each other.

Definition 2 *A pair (X, Y), where $X \subseteq U$, $Y \subseteq V$, is called a formal concept in the context (U, V, R) if $X = Y^*$ and $Y = X^*$. X is referred to as the extension, and Y is referred to as the intension, of the concept (X, Y).*

There are some alternative representation of a concept. A pair (X, X^*) that satisfies $X \subseteq U$ and $X = X^{**}$ is a formal concept. A pair (Y^*, Y) satisfying $Y \subseteq V$ and $Y = Y^{**}$ is also a formal concept.

The family of all formal concepts derived from a formal context can be constructed as a complete lattice, called *concept lattice*. Figure 1 illustrates the concept lattice for the formal context of Table 1.

The meet and join of the formal concepts in the lattice are characterized by the following operations [7, 16].

Theorem 1 *The formal concept lattice is a complete lattice in which the meet and join are given by:*

$$\bigwedge_{t \in T} (X_t, Y_t) = (\bigcap_{t \in T} X_t, (\bigcup_{t \in T} Y_t)^{**}),$$

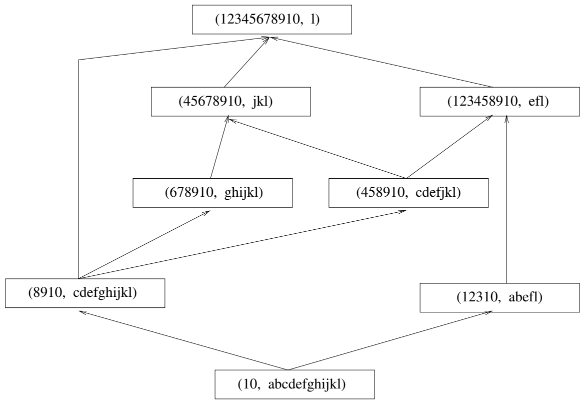

Figure 1. Formal Concept Lattice for the Table 1

$$\bigvee_{t \in T} (X_t, Y_t) = ((\bigcup_{t \in T} X_t)^{**}, \bigcap_{t \in T} Y_t),$$

where T is an index set, and for every $t \in T$, (X_t, Y_t) is a formal concept.

The order relation of concepts in the lattice can be defined based on the set inclusion relation [20].

Definition 3 *For two formal concepts (X_1, Y_1) and (X_2, Y_2), (X_1, Y_1) is a sub-concept of (X_2, Y_2), written $(X_1, Y_1) \preceq (X_2, Y_2)$, and (X_2, Y_2) is a super-concept of (X_1, Y_1), if and only if $X_1 \subseteq X_2$, or equivalently, if and only if $Y_2 \subseteq Y_1$.*

It is easy to know that a more general concept is represented by a larger set of objects that share a smaller set of attributes. Conversely, a more specific concept has a smaller set of objects and a larger set of attributes.

4 Hierarchical Class Analysis

Hierarchical class analysis is proposed to describe and summarize the data by using set-theoretical relations among objects and attributes [2]. Based on those relations, the objects and attributes can be grouped into classes, and then, hierarchies of classes can be constructed. In this section, we formally study hierarchical class analysis based on set-theoretical formulations.

4.1 Object and Attribute Classes

With a formal context, objects and attributes can describe each other. For an object x, the attributes xR are semantically considered as the characteristics or features of the object. Accordingly, objects are viewed as equivalent if they have a same set of attributes. Therefore, one can define an equivalence relation on objects based on the attributes.

Let $E_U \subseteq U \times U$ denote a binary relation on the universe of objects U defined by:

$$x \; E_U \; x' \Longleftrightarrow xR = x'R.$$

It is in fact an equivalence relation. The equivalent objects can be grouped together into a set, called an *object class*, and is denoted by $[x]$ [11]. An object class including the object x is defined by:

$$\begin{aligned} [x] &= \{x' \in U \mid x' \; E_U \; x\}, \\ &= \{x' \in U \mid xR = x'R\}. \end{aligned}$$

Another view to look at the equivalence classes is that the equivalence relation E_U partitions the universe of objects U into disjoint subsets, which are the object classes [11, 12].

Elements of an object class are indiscernible. Each object class is considered as a whole instead of individuals inside [12].

The partition of the universe U is the family of all object classes and commonly known as the *quotient set* U/E_U of U induced by E_U.

In the same way, attribute classes on the universe V is given by:

$$\begin{aligned} [y] &= \{y' \in V \mid y' \; E_V \; y\}, \\ &= \{y' \in V \mid Ry' = Ry\}. \end{aligned}$$

The universes U and V can be expressed as the unions of object and attribute classes.

By considering the sufficiency operator *, one can know that the set $[x]^* = xR$ is the set of attributes possessed by objects in class $[x]$, and the set $[y]^* = Ry$ is the set of objects having the attribute y. Furthermore, an object class $[x]$ is included in the object set $[x]^{**}$, and an attribute class $[y]$ is include in the attribute set $[y]^{**}$.

Object and attribute classes can also be defined by basic set assignments between U and V as follows [19]:

Definition 4 *For a set of objects $X \subseteq U$, a basic set assignment $^b : 2^U \longrightarrow 2^V$ is defined by:*

$$X^b = \{y \in V \mid Ry = X\}.$$

The following set:

$$\{X^b \neq \emptyset \mid X \subseteq U\}$$

is the family of attribute classes.

Definition 5 *For a set of attributes $Y \subseteq V$, a basic set assignment and $^b : 2^V \longrightarrow 2^U$ is defined by:*

$$Y^b = \{x \in U \mid xR = Y\}.$$

The following set:

$$\{Y^b \neq \emptyset \mid Y \subseteq V\}$$

is the family of object classes.

The relationships of basic set assignment and object and attribute classes can be presented as $(xR)^b = [x]$ and $(Ry)^b = [y]$.

The basic set assignments explore a type of association between an object class and a set of attributes, or an attribute class and a set of objects. That is, an attribute class uniquely associates with a particular set of objects, and an object class uniquely associates with a particular set of attributes.

Example 1 *In Table 1, the object classes are:*

$$\begin{aligned} Y_1^b &= [1] &&= \{1,2,3\}, \\ Y_2^b &= [4] &&= \{4,5\}, \\ Y_3^b &= [6] &&= \{6,7\}, \\ Y_4^b &= [9] &&= \{8,9\}, \\ Y_5^b &= [10] &&= \{10\}. \end{aligned}$$

The attribute classes are:

$$\begin{aligned} X_1^b &= [a] &&= \{a,b\}, \\ X_2^b &= [c] &&= \{c,d\}, \\ X_3^b &= [e] &&= \{e,f\}, \\ X_4^b &= [g] &&= \{g,h,i\}, \\ X_5^b &= [j] &&= \{j,k\}, \\ X_6^b &= [l] &&= \{l\}. \end{aligned}$$

The sets of attributes that associate with object classes are:

$$\begin{aligned} Y_1 &= \{[a],[e],[l]\}, \\ Y_2 &= \{[c],[e],[j],[l]\}, \\ Y_3 &= \{[g],[j],[l]\}, \\ Y_4 &= \{[c],[e],[g],[j],[l]\}, \\ Y_5 &= \{[a],[c],[e],[g],[j],[l]\}. \end{aligned}$$

The sets of objects that associate with attribute classes are:

$$\begin{aligned} X_1 &= \{[1],[10]\}, \\ X_2 &= \{[4],[8],[10]\}, \\ X_3 &= \{[1],[4],[8],[10]\}, \\ X_4 &= \{[6],[8],[10]\}, \\ X_5 &= \{[4],[6],[8],[10]\}, \\ X_6 &= \{[1],[4],[6],[8],[10]\}. \end{aligned}$$

	[a]	[c]	[e]	[g]	[j]	[l]
[1]	1	0	1	0	0	1
[4]	0	1	1	0	1	1
[6]	0	0	0	1	1	1
[8]	0	1	1	1	1	1
[10]	1	1	1	1	1	1

Table 2. The reduction of Table 1

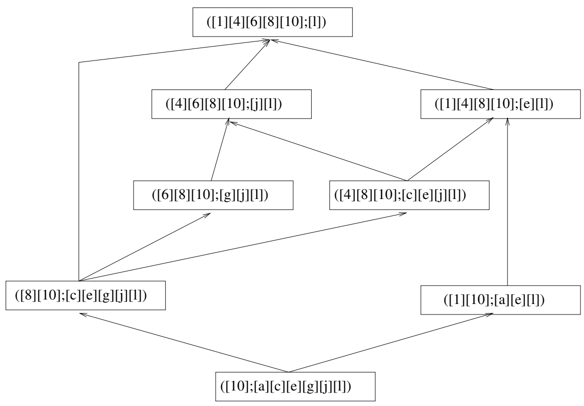

Figure 2. Formal concept lattice induced from the Table 2

By using the equivalence classes , a reduction of Table 1 is given by Table 2.

Moreover, the concept lattice derived from Table 1 can be re-constructed by using equivalence classes and is given by Figure 2.

In terms of the basic set assignment, we can re-express operation * as follow [19]:

$$X^* = \bigcup \{A^b \mid A \subseteq U, X \subseteq A\},$$
$$Y^* = \bigcup \{B^b \mid B \subseteq V, Y \subseteq B\}.$$

This formulation presents the connection between basic set assignment ^{b} and the modal-style operator *. Moreover, by combing the two operators, one can easily have:

$$X^{b*} = X, \text{ where } X^b \neq \emptyset, X \subseteq U,$$
$$Y^{b*} = Y, \text{ where } Y^b \neq \emptyset, Y \subseteq V.$$

A class may be considered as a basic definable unit in the universes [12, 20]. In other words, under an equivalence relation, a class is the smallest non-empty observable, measurable, or definable subset of 2^U or 2^V. A union of some classes is called a composed class [11, 12]. By extending the definability of equivalence classes, we assume that a union of some classes is also definable. By adding the empty set $\emptyset$, a new family of all composed classes $\sigma(U/E_U)$, which is a subsystem of 2^U, can be constructed. $\sigma(U/E_U)$ is closed under set complement, intersection, and union. Obviously, a subsystem $\sigma(V/E_V)$ of 2^V can be developed based on the universe V and the equivalence relation E_V. Rough set approximations can be defined based on the subsystem $\sigma(U/E_U)$ [12, 20].

4.2 Hierarchies of Classes

Based on notation xR, there exists an order relation, denoted as $\sqsubseteq$, among the object classes.

Given two object classes $[x]$ and $[x']$ on U, an order relation $\sqsubseteq$ between them is given by:

$$[x] \sqsubseteq [x'] \iff xR \subseteq x'R.$$

$[x]$ is called a sub-class of $[x']$. Conversely, $[x']$ is called a super-class of $[x]$. In fact, we order the object classes by using their associated set of attributes. Similarly, for two attribute classes $[y]$ and $[y']$ on V, $[y]$ is a sub-class of $[y']$ and $[y']$ is a super-class of $[y]$ if

$$[y] \sqsubseteq [y'] \iff Ry \subseteq Ry'.$$

According to the order relation between the classes, a hierarchical structure of the classes for a formal context can be constructed. For example, hierarchies of classes for Table 2 is given by Figure 3. The hierarchy of attribute classes in the part (b) of Figure 3 is presented upside down in the part (c). The hierarchies of object classes in the part (a) and attribute classes in the part (b) are combined together by linking the bottom classes of two hierarchies with dot lines.

From the observation in the Figure 3, we know that there exists associations between object classes and attribute classes [2]. If an attribute class is included in the set of attributes possessed by an object class, that is, $[y] \subseteq xR$, we say that the object class $[x]$ associates with the attribute class $[y]$. This type of association is symmetric. That means if an object class associates with an attribute class, then the attribute class must associate with the object class.

Furthermore, if an object class associates with an attribute class, it also associates with all super-classes of the attribute class. That is, an object class should associates with a union of attribute classes. Likewise, an attribute class associates with a union of object classes, namely, an object class and all its the super-classes.

4.3 Basic Sets of Objects and Attributes

Based on the basic set assignments, each object class must uniquely associate a set of attributes, and each

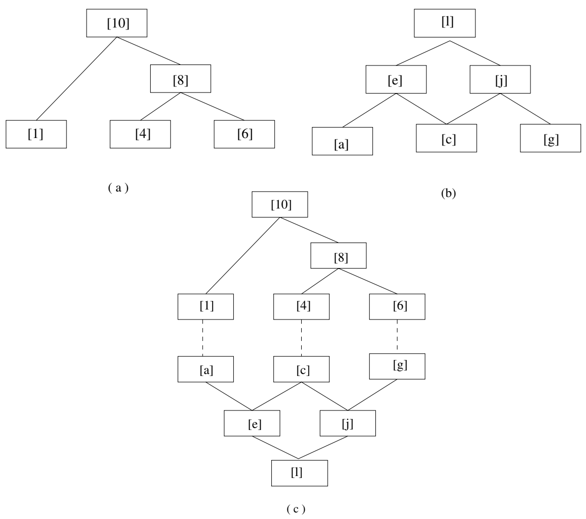

Figure 3. Hierarchies of Classes for the Table 1

attribute class must uniquely associate a set of objects. In fact, those object and attribute sets are the basic sets that not only reflect the equivalence relation and order relation, but also can be used to form the concepts derived from the formal context.

Let AT denote the family of all sets of attributes that each associates with an object class, that is,

$$AT = \{Y \mid Y \subseteq V, Y^b \neq \emptyset\}.$$

Let OB denote the family of all sets of objects that each associates with an attribute class, that is,

$$OB = \{X \mid X \subseteq U, X^b \neq \emptyset\}.$$

By the definition of object classes, one knows that the union of elements in OB is the universe of objects, that is, $U = \bigcup\{A \mid A \in OB\}$. Similarly, the union of elements in AT is the universe of attributes, that is, $V = \bigcup\{B \mid B \in AT\}$.

Each element in AT is a set of attributes that uniquely associates with an object class. Thus, the objects that associate with the same set of attributes in AT are viewed as equivalent and grouped into an object class. Furthermore, the order among the elements in AT reflects the order relation among the object classes. Apparently, the elements in OB can be used to define the equivalence relation among attributes in an attribute class and the order relation among the attribute classes.

By adding the universe U into the set OB and the universe V into the set AT, we can have following theorems.

Theorem 2 *For a $X \in OB$, $X^{**} = X$. For a $Y \in AT$, $Y^{**} = Y$.*

This theorem means that the pair (X, X^*) is a formal concept. Since the object set $X \in OB$ associates with an attribute class, so we know that $X^{b*} = X$. By considering the Property (3) of the sufficiency operator *, we can have that $X = X^{b*} = X^{b***} = X^{**}$. Thus, based on formal concept analysis, the pair (X, X^*) that satisfies $X = X^{**}$ is a formal concept. In the same way, the pair (Y^*, Y) that satisfies $Y^{**} = Y$ is also a formal concept.

Furthermore, by considering the relationship between basic set assignments ^{b} and operator *, all formal concepts derived from a formal context can be formed by the elements in OB or AT. That is, we can have following theorem.

Theorem 3 *For a formal concept (X, Y), we have:*

$$X = \bigcap\{A \mid A \in OB, X \subseteq A\},$$
$$Y = \bigcap\{B \mid B \in AT, Y \subseteq B\}.$$

Suppose there exists a formal concept (X, Y) such that $X \neq \bigcap\{A \mid A \in OB, X \subseteq A\}$. Since $X^* = \bigcup\{A^b \mid A \in OB, X \subseteq A\}$, so we can have $X^{**} = \bigcap\{A \mid A \in OB, X \subseteq A\}$. Then, we can have $X^{**} \neq X$. This result is contradiction because the pair (X, Y) is a formal concept and $X^{**} = X$. Thus, a formal concept can be formed based on the elements in OB or AT.

Since the order relation among the elements in OB (AT) is defined by set inclusion, some elements are the subset of other elements. In other words, some elements are the union of other elements. Each elements in OB (AT) must associate with an equivalence class. Therefore, the elements that are the union of other elements must associate with super-classes. The object (attribute) sets that are used to form other object (attribute) sets in OB (AT) may be viewed as the unit set.

Let $\mathcal{BO}$ denote all the object sets in OB that are not the union of other object sets in OB.

Definition 6 *In a formal context, an object set $X \in OB$ is called an object block if*

$$X \neq \bigcup\{A \mid A \in OB, A \neq X, A^b \sqsubseteq X^b\}.$$

$\mathcal{BO}$ is the family of all object blocks.

Let $\mathcal{BA}$ denote all the attribute sets in AT that are not the union of other attribute sets in AT.

Definition 7 *In a formal context, an attribute set $Y \in AT$ is called an attribute block if*

$$Y \neq \bigcup\{B \mid B \in AT, B \neq Y, B^b \sqsubseteq Y^b\}.$$

$\mathcal{BA}$ is the family of all attribute blocks.

The union of all the object sets in $\mathcal{BO}$ is the universe of objects U. That is, $U = \bigcup\{A \mid A \in \mathcal{BO}\}$. The union of all the attribute sets in $\mathcal{BO}$ is the universe of attributes V. That is, $V = \bigcup\{B \mid B \in \mathcal{BA}\}$.

Each element in $\mathcal{BO}$ ($\mathcal{BA}$) must be in OB (AT). Thus, for $X \in \mathcal{BO}$, the pair (X, X^*) is a formal concept. For $Y \in \mathcal{BA}$, the pair (Y^*, Y) is also a formal concept.

Furthermore, all formal concepts within a formal context can be formed by the elements in $\mathcal{BO}$ or $\mathcal{BA}$.

Theorem 4 *For a formal concept (X, Y), we have:*

$$X = \bigcap\{A \mid A \in \mathcal{BO}, X \subseteq A\},$$

or

$$Y = \bigcap\{B \mid B \in \mathcal{BA}, Y \subseteq B\}.$$

Example 2 *In Table 2, the object blocks and attribute blocks are:*

$$\begin{array}{ll} X_1 = \{[1],[10]\}, & Y_1 = \{[a],[e],[l]\}, \\ X_2 = \{[4],[8],[10]\}, & Y_2 = \{[c],[e],[j],[l]\}, \\ X_3 = \{[6],[8],[10]\}, & Y_3 = \{[g],[j],[l]\}. \end{array}$$

We know that $U = X_1 \cup X_2 \cup X_3$ and $V = Y_1 \cup Y_2 \cup Y_3$.

By considering the sufficiency operator *, we know that $X_1 = X_1^{**}$. So, the pair (X_1, X_1^*) is a formal concept.

For the object class [1]*, it associates with the set of attributes* $\{[a],[e],[l]\} = \{Y_1\}$*. For the object class* [10]*, it associates with the whole universe* $V = \{[a],[c],[e],[g],[j],[l]\} = \{Y_1, Y_2, Y_3\}$*. Therefore, the object class* [10] *is a super-class of the object class* [1] *because the attribute blocks for* [10] *includes the attribute blocks for* [1]*.*

For the formal concept

$$(\{[8],[10]\}, \{[c],[e],[g],[j],[l]\}),$$

in Figure 2, we can have:

$$\begin{aligned} \{[8],[10]\} &= \{[4],[8],[10]\} \bigcap \{[6],[8],[10]\} \\ &= X_2 \bigcap X_3. \end{aligned}$$

The family of object blocks $\mathcal{BO}$ and the family of attribute blocks $\mathcal{BA}$ have similar properties. However, the sizes of them are not necessarily equal. In fact, when every object block associates with an attribute class that has no sub-class, and every attribute block associates with an object class that has no sub-class, the sizes of $\mathcal{BO}$ and $\mathcal{BA}$ may be equal.

5 Conclusion

In this paper, formal set-theoretical formulations of hierarchical class analysis is presented. The equivalence relation, order relation, and association between objects and attributes in hierarchical class analysis are formulated and discussed. Moreover, formal concept analysis based on hierarchical class analysis is provided. A family of basic sets of objects and attributes formed in hierarchical class analysis can be used to construct the formal concepts and concept lattices.

Some research topics, such as finding unique object and attribute blocks, in hierarchical class analysis are not studied in this paper. Further studies of the connections between hierarchical class analysis and formal concept analysis need to be done in the future research.

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