Rough Sets and Three-Way Decisions

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Abstract. The notion of three-way decisions was originally introduced by the needs to explain the three regions of probabilistic rough sets. Recent studies show that rough set theory is only one of possible ways to construct three regions. A more general theory of three-way decisions has been proposed, embracing ideas from rough sets, interval sets, shadowed sets, three-way approximations of fuzzy sets, orthopairs, square of oppositions, and others. This paper presents a trisecting-and-acting framework of three-way decisions. With respect to trisecting, we divide a universal set into three regions. With respect to acting, we design most effective strategies for processing the three regions. The identification and explicit investigation of different strategies for different regions are a distinguishing feature of three-way decisions.

1 Introduction

In rough set theory [21, 23], there exist two representations of approximations of an undefinable set by definable sets [38], namely, a pair of lower and upper approximations or three pair-wise disjoint positive, boundary and negative regions. Although the two representations are mathematically equivalent, they provide different hints when we attempt to generalize rough set theory or to study its relationships to other theories. In fact, the two representations have led to two distinct research directions and two useful generalizations of rough set theory.

The representation with a pair lower and upper approximations was used to establish a close connection between rough set theory and modal logics [20, 41]. More specifically, the lower and upper approximation operators in rough set theory are interpreted, respectively, in terms of the necessity and possibility operators in modal logics [1]. This connection immediately suggests generalized rough set models induced by various classes of non-equivalence relations [41]. The representation with three regions motivated the introduction of a theory of three-way decisions (3WD)[1] [33–35]. The three regions are interpreted in terms

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[1] TWD was first used as an abbreviation for three-way decisions. Hu [10] suggested 3WD, which seems to be a better abbreviation as we can abbreviate two-way decisions as 2WD.

of three types of classification rules: rules of acceptance, deferment and rejection for the positive, boundary and negative regions, respectively [10, 13, 15, 16, 19, 30, 33, 34]. The theory of three-way decision opens new avenues to extend rough set research. In fact, in a recent scientometrics study of rough sets research in the past three decades, JT Yao and Zhang concluded [31], "The theory of three-way decisions, motivated by rough set three-regions but goes beyond rough sets, is a promising research direction that may lead to new breakthrough."

On the one hand, the theory of three-way decisions adopts three pair-wise disjoint regions from rough set theory as one of its basic notions. On the other hand, it moves beyond rough set theory in the sense that the latter is only one of many possible ways to construct and to use three regions. Other theories involving three regions include, for example, interval sets [32], three-way approximations of fuzzy sets [7, 46], shadowed sets [24], orthopairs (i.e., a pair of disjoint sets) [2–4], and squares of oppositions [5, 8, 36]. The theory of three-way decisions embraces ideas from these theories and, at the same time, introduces its own notions and concepts.

There is a fast growing interest in the theory of three-way decisions, resulting in several edited books [11, 17, 44] and extensive research results (for example, see a sample of papers published in 2015 [12, 14, 15, 18, 25–30, 45, 48, 49] and in the session on three-way decisions in this volume). By generalizing results reported in [33, 35, 43], this paper presents a trisecting-and-acting framework of three-way decisions. The trisecting and acting reflect two important and fundamental components of three-way decisions.

2 Two Representations of Rough Set Approximations

Let U denote a finite non-empty universal set of objects and E denote an equivalence relation on U. The pair $apr = (U, E)$ is called a Pawlak approximation space [21]. The equivalence class containing an object $x \in U$ is given by $[x]_E = [x] = \{y \in U \mid xEy\}$, where the subscript E represents the equivalence relation E and is dropped when no confusion arises. A fundamental notion of rough set theory is the approximation of a subset of U by using the family of equivalence classes $U/E = \{[x] \mid x \in U\}$.

There exist two formulations for defining rough set approximations of a subset of objects $X \subseteq U$. One way to define rough set approximations is by using a pair of lower and upper approximations [21, 23]:

$$\begin{aligned} \underline{apr}(X) &= \{x \in U \mid [x] \subseteq X\}, \\ \overline{apr}(X) &= \{x \in U \mid [x] \cap X \neq \emptyset\} = \{x \in U \mid [x] \nsubseteq X^c\}, \end{aligned} \tag{1}$$

where $X^c = U - X$ denotes the complement of X. By definition, it follows that $\underline{apr}(X) \subseteq X \subseteq \overline{apr}(X)$, namely, X lies between its lower and upper approximations. Thus, this definition has a very appealing physical interpretation. Another way is to define rough set approximations by using three pair-wise disjoint positive, negative and boundary regions [40]:

$$\mathrm{POS}(X) = \{x \in U \mid [x] \subseteq X\},$$

$$\begin{aligned}\mathrm{NEG}(X) &= \{x \in U \mid [x] \subseteq X^c\},\\ \mathrm{BND}(X) &= \{x \in U \mid [x] \not\subseteq X \wedge [x] \not\subseteq X^c\}.\end{aligned} \tag{2}$$

By definition, the three regions are pair-wise disjoint and their union is the universe U. Since some of the regions may be empty, the three regions do not necessarily form a partition of the universe. In this paper, we call the family of three regions a tripartition, or a trisection, of the universe by slightly abusing the notion of a partition. Given any pair of two regions, we can derived the third one through set complement. Thus, the notion of tree regions is related to the notion of orthopairs [2, 3], namely, a pair of disjoint sets.

The definition based on a pair of lower and upper approximations has been widely used in the main stream research. Unfortunately, the three regions were treated as a derived notion defined in terms of the lower and upper approximations [21]. By looking at Equations (1) and (2), we can conclude that the two formulations can, in fact, be developed independent of each other. In addition, the two formulations are related to each other as follows:

$$\begin{aligned}\mathrm{POS}(X) &= \underline{apr}(X),\\ \mathrm{NEG}(X) &= (\overline{apr}(X))^c,\\ \mathrm{BND}(X) &= \overline{apr}(X) - \underline{apr}(X),\end{aligned} \tag{3}$$

and

$$\begin{aligned}\underline{apr}(X) &= \mathrm{POS}(X),\\ \overline{apr}(X) &= \mathrm{POS}(X) \cup \mathrm{BND}(X).\end{aligned} \tag{4}$$

Thus, one can formulate rough set theory by using either a pair of approximations or three regions as its primitive notions.

A salient of definitions given by Equations (1) and (2) is that approximations are uniformly defined in terms of the set-inclusion relation $\subseteq$. This is obtained at the expenses of using both X and and its complement X^c. Equations (1) and (2) are qualitative rough set approximations. An advantage of the definitions using only the inclusion relation $\subseteq$ is that we can generalize qualitative approximations into quantitative approximations by considering a certain degree of inclusion [40]. Probabilistic rough sets, in particular, decision-theoretic rough sets [39, 42], are examples of quantitative generalizations with three probabilistic positive, boundary and negative regions.

The notion of three-way decisions is introduced to meet the needs to explain three probabilistic regions [33, 34]. The definition of approximations in the three-region-based formulation of rough set theory provides hints on building models of three-way classification [6, 9, 22, 34]. The theory of three-way decisions adopts the notion of three regions from rough set theory as one of its basic notions.

3 Refining Formulations of Three-Way Decisions

Research on three-way decisions had been focused mainly on the division of a universal set into three pair-wise disjoint regions. Yao and Yu [43] argued that one must design different strategies for processing the three regions. In this section, we introduce a trisecting-and-acting framework. Within the framework, we examine the three main stages in the evolution of three-way decisions, moving from specific to more general models.

3.1 Probabilistic Three-Way Classifications

The first attempt to formulate three-way decisions is based on a new interpretation of probabilistic rough sets for three-way classification [33]. We assume that the conditional probability $Pr(X|[x])$ denotes the degree to which $[x]$ is a subset of X and $Pr(X^c|[x]) = 1 - Pr(X|[x])$ denotes the degree to which $[x]$ is a subset of X^c. One of the results of the decision-theoretic rough set model [39, 42] is three probabilistic positive, boundary and negative regions defined by: for a pair of thresholds (α, β) with $0 \leq \beta < \alpha \leq 1$,

$$\begin{aligned} \mathrm{POS}_{(\alpha,\beta)}(X) &= \{x \in U \mid Pr(X|[x]) \geq \alpha\} \\ &= \{x \in U \mid Pr(X^c|[x]) \leq 1 - \alpha\}, \\ \mathrm{BND}_{(\alpha,\beta)}(X) &= \{x \in U \mid \beta < Pr(X|[x]) < \alpha\}, \\ \mathrm{NEG}_{(\alpha,\beta)}(X) &= \{x \in U \mid Pr(X|[x]) \leq \beta\} \\ &= \{x \in U \mid Pr(X^c|[x]) \geq 1 - \beta\}. \end{aligned} \tag{5}$$

The Pawlak rough set model is a special case in which $\alpha = 1$ and $\beta = 0$.

While Pawlak positive and negative regions do not have classification error, probabilistic positive and negative regions have classification errors. This observation calls for a new interpretation of probabilistic regions. Yao [33, 34] introduces the notion of three-way decisions for such a purpose. Consider an arbitrary object $y \in [x]$. We make one of the following three decisions:

- If $Pr(X|[x]) \geq \alpha$, y has a high probability at or above α to be in X. We accept y being an object in X, with an understanding that this acceptance is associated with an error rate of $Pr(X^c|[x]) \leq 1 - \alpha$.
- If $Pr(X|[x]) \leq \beta$, or equivalently, $Pr(X^c|[x]) \geq 1-\beta$, y has a low probability at or below β to be in X. We reject y being an object in X, with an understanding that this rejection is associated with an error rate of $Pr(X|[x]) \leq \beta$.
- If $\beta < Pr(X|[x]) < \alpha$, the probability of y in X is neither high nor low, but in the middle. We can neither accept nor reject y being an object in X. In this case, we make a noncomittment decision.

The pair of thresholds is related to the tolerant levels of errors for acceptance and rejection. The three actions are meaningful. In the light of the trisecting-and-acting framework, Fig. 1 illustrates the two basic components of a three-way classification model. Equation (5) achieves the goal of trisecting U. Decisions of acceptance, noncommitment or rejection are strategies for the three regions.

One can observe that three-way classifications directly borrow notions, concepts and terminology from rough set theory. In the context of classification,

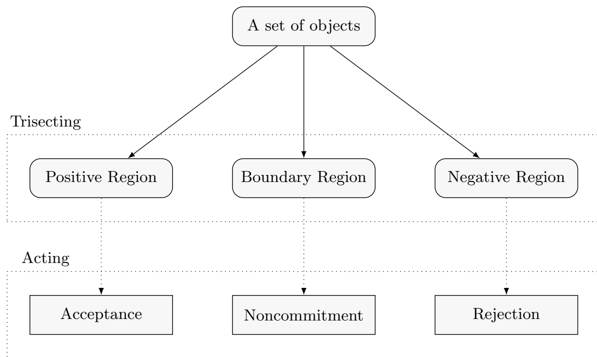

Fig. 1. Three-way classification

the physical meaning of positive, boundary and negative regions, and the associated decisions of acceptance, noncommitment and rejection, is meaningful. Unfortunately, these notions may not be appropriate to describe other types of three-way decisions. New formulations of three-way decisions are required.

3.2 Evaluation-based Three-Way Decisions

The second attempt to formulate three-way decisions is based on an evaluation function and a description of three regions in generic terms. The result is an evaluation-based model of three-way decisions [35].

Suppose $(\mathbf{L}, \preceq)$ a totally ordered set. For two objects $x, y \in U$, if $x \preceq y$, we also write $y \succeq x$. Suppose $v : U \longrightarrow \mathbf{L}$ is an evaluation function. For an object $x \in U$, $v(x)$ is its evaluation status value (ESV). Based on their evaluation status values, one can arrange objects in U in an increasing order: objects with lower values to the left and objects with larger values to the right. By using a pair of thresholds (α, β) from $\mathbf{L}$ with $\beta \prec \alpha$ (i.e., $\beta \preceq \alpha$ and $\neg(\alpha \preceq \beta)$), one can divide U into three regions [35]:

$$\begin{aligned} \mathrm{L}(v) &= \{x \in U \mid v(x) \succeq \alpha\}, \\ \mathrm{M}(v) &= \{x \in U \mid \beta \prec v(x) \prec \alpha\}, \\ \mathrm{R}(v) &= \{x \in U \mid v(x) \preceq \alpha\}. \end{aligned} \tag{6}$$

To be consistent with the increasing ordering of objects from left to right according to their evaluation status values, Yao and Yu [43] call the three regions the left, middle, and right regions, respectively, or simply, L-region, M-region, and R-region. Under the condition $\beta \prec \alpha$, the three regions are pair-wise disjoint and their union is the universe U.

Fig. 2 depicts an evaluation-based three-way decision model. We divide the universe U according to Equation (6). We design different strategies to process the three regions. By using more generic labels and terms in description, we

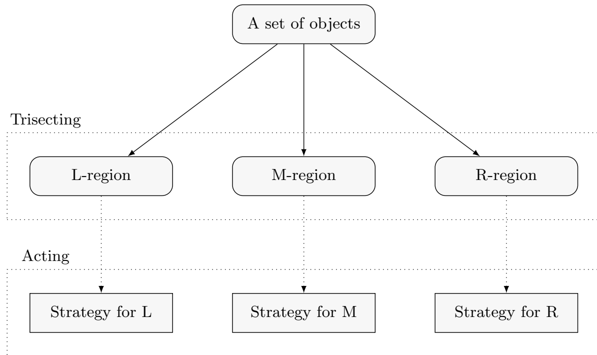

Fig. 2. Evaluation-based three-way decisions

have more general model. A three-way classification model may be viewed as a special case of evaluation-based three-way decisions. More specifically, in a three-way classification model, the totally ordered set $(\mathbf{L}, \preceq)$ is $([0,1], \leq)$, the evaluation function is given by the conditional probability, $v(x) = Pr(X|[x])$, the L-region, M-region, and R-region are, respectively, the positive, boundary and negative regions, and the strategies for L-, M-, and R-regions are the decisions of acceptance, noncommitment and rejection, respectively.

There still exist problems with the second formulation of three-way decisions. By using an totally ordered set $(\mathbf{L}, \preceq)$, we, in fact, impose an ordering on the three regions, as indicated by the names of left, middle and right regions. Such an ordering may unnecessarily limit the generality of three-way decisions. In some situations, it may be difficult to give an analytic formula for computing the evaluation status values of objects. Therefore, we need to further search for new formulations of three-way decisions.

3.3 A Trisecting-and-acting Framework of Three-Way Decisions

In this paper, we present a third attempt to formulate three-way decisions. The main objective is to avoid limitations of the first two attempts and to describe three-way decisions in even more generic terms.

To avoid an ordering of three regions, we directly consider a division of U into three parts without explicitly referring to an evaluation function. That is, one of the primitive notions is an abstract method to divide U into three pair-wise disjoint regions. This can be defined by a function,

$$\tau : U \longrightarrow \mathbf{T}, \tag{7}$$

where $\mathbf{T}$ is a set with three values, which each value for each of the three regions. We are not concerned with any particular detail procedure, nor are we concerned with the reasons that lead to the three regions. Consider election as an example.

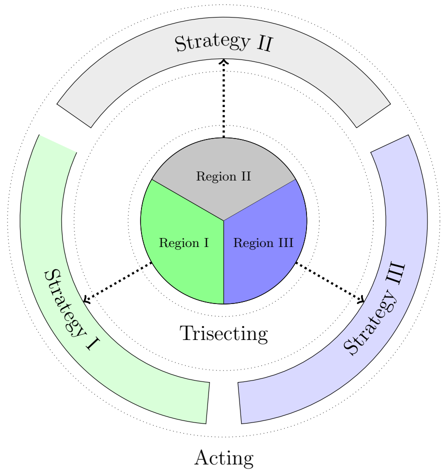

Fig. 3. A conceptual model of three-way decisions

By surveying voters about their intended voting decisions, we may divide a set of surveyed voters into three regions: those who support a candidate, those who are undecided or are unwilling to tell their decisions, and those who oppose the candidate. Such a division is obtained without an evaluation function that explains voters' decisions. Given a division, we must work on strategies for processing each of the three regions. Consider again the example of election. Strategies may aim at retaining supporters and transforming those who are undecided or oppose the candidate into supporters.

Without loss of generality, we assume $\mathbf{T} = \{\mathbf{i}, \mathbf{ii}, \mathbf{iii}\}$. According to the values of τ, we can divide the universe U into three pair-wise disjoint regions:

$$\begin{aligned} \mathbb{I}(\tau) &= \{x \in U \mid \tau(x) = \mathbf{i}\}, \\ \mathbb{II}(\tau) &= \{x \in U \mid \tau(x) = \mathbf{ii}\}, \\ \mathbb{III}(\tau) &= \{x \in U \mid \tau(x) = \mathbf{iii}\}. \end{aligned} \tag{8}$$

We immediately have $\mathbb{I}(\tau) \cup \mathbb{II}(\tau) \cup \mathbb{III}(\tau) = U$, $\mathbb{I}(\tau) \cap \mathbb{II}(\tau) = \emptyset$, $\mathbb{I}(\tau) \cap \mathbb{III}(\tau) = \emptyset$, and $\mathbb{II}(\tau) \cap \mathbb{III}(\tau) = \emptyset$. To pictorially illustrate an unordered three regions, it is impossible to draw them linearly as in the first two formulations. Instead, we can draw them in a circle so that any region contacts with other two regions, as shown in Fig. 3. In this way, we no longer have a positional naming system used in the evaluation-based model. Instead, we refer to the three regions as regions I, II, and III, respectively. Corresponding to the three regions, we have strategies I, II, and III, respectively. It should be pointed out that we use I, II, and III simply as different labels without considering their numeric values.

With the new naming system, we avoid the narrow meaning that is implicitly suggested by the two naming systems of a three-way classification model and an evaluation-based model. All three regions are treated on the equal footing, without any preference of one to another. As shown by Fig. 3, any region is connected to the other two regions. No region takes a better position than other two regions. Such an understanding is consistent with a fact that for different applications we may focus on any one, two or all of the three regions.

4 Future Research in Three-way Decisions

At a first look, the three attempts to formulate three-way decisions are characterized by changes of naming systems and pictorial descriptions. It is important to realize that these changes in fact enable us to gain more insights into three-way decisions. We are moving from concrete levels of specific models to a more abstract level of a general framework of three-way decisions.

The dividing-and-acting framework, as shown by Fig. 3, provides a high-level abstract description of three-way decisions and opens new avenues for further studies. This conceptual model is very useful for us to explain the basic ideas and components of three-way decisions. In order to realize ideas in the conceptual model, we need to build computational models of three-way decisions. By considering different ways to divide a universe and different strategies for processing three regions, we can derive many concrete computational models of three-way decisions.

An explicit separation of the two tasks of three-way decisions, namely, trisecting and acting, is mainly for the purpose of building an easy-to-understand model. In many situations, the two tasks in fact weave together as one and cannot be easily separated. A meaningful trisection of U depends on the strategies and actions used for processing the three regions. Effective strategies and actions of processing depend on an appropriate trisection of U. The two are just different sides of the same coin; we cannot have one without a consideration of the other. It is desirable that we both separate and integrate the tasks of trisecting and acting, depending on particular problems.

We can pursue further studies of three-way decisions by focusing on the two components of the dividing-and-acting framework. With respect to dividing, we suggest the following topics:

- Methods for trisecting a universe: In addition to evaluation-based methods, we may investigate other approaches. One possible is to construct a trisection through a pair of bisections. Another possibility is to consider a ranking of objects, which is a fundamental notion in theories of decision-making. Three regions correspond to the top, middle, and bottom segments of the ranking. A third possibility is to seek for a statistical interpretation of a trisection.
- Different classes of evaluation functions: Probabilistic three-way classifications use probability as an evaluation function. In general, we may consider other types of evaluation functions, including, for example, possibility functions, fuzzy membership functions, Bayesian confirmation measures, similarity measures, and subsethood measures. It is necessary to interpret an evaluation function in operable notions.

- Methods for determining the pair of thresholds: For an evaluation-based model, we need to investigate ways to compute and to interpret a pair of thresholds. An optimization framework can be designed to achieve such a goal. That is, a pair of thresholds should induce a trisection that optimizes a given objective function. By designing different objective functions for different applications, we gain a great flexibility.

A meaningful trisection depends on the strategies and actions for processing three regions. With respect to acting, we suggest the following topics:

- Descriptive rules for three regions: To fully understand three regions and to design effective strategies and actions for processing them, as a prerequisite, we must be able to describe and to represent the three regions. Descriptive rules summary the main features of the three regions, with each rule characterizes a portion of a specific region.
- Predictive rules for three regions: We can also construct predictive rules from the three regions to make decisions for new instances.
- Actionable rules for transferring objects between regions: In some situations, it is desirable to transfer objects from one region to another. We can use actionable rules in order to make such a move possible.
- Effective strategies and actions for processing three regions: For different regions, we must design the most suitable and effective strategies for processing. It may be sufficient to focus on one of the three regions. It may also happen that we must consider two or three regions simultaneously.
- Comparative studies: We can perform comparative studies by considering a pair regions together. In so doing, we can identify the differences between, and similarities of, the two regions. Descriptive rules may play a role in such comparative studies. The results of comparative study may also lead to actionable rules for transferring objects between regions.

In designing strategies for processing one region, we may have to consider the strategies for processing other regions. Interactions of strategies for different regions may play an important role in achieving a best trisection of the universe.

To fully understand and appreciate the value and power of three-way decisions, we must look into fields where the ideas of three-way decisions have been used either explicitly or implicitly. In one direction, one can adopt ideas from these fields to three-way decisions. In the other direction, we can apply new results of three-way decisions to these fields. We can cast a study of three-way decisions in a wider context and extend applications of three-way decisions across many fields. In the next few years, we may investigate three-way decisions in relation to interval sets [32], three-way approximations of fuzzy sets [7, 46], shadowed sets [24], orthopairs [2–4], squares of oppositions [5, 8, 36], granular computing [37], and multilevel decision-making [47].

5 Conclusion

Although three-way decisions were originally motivated by the needs to properly interpret three regions of probabilistic rough sets, extensive studies in the last few years have moved far beyond. By realizing that the notion of three regions has been used in many fields, we can formulate a more general theory of three-way decisions. Ideas from rough sets, interval sets, three-way approximations of fuzzy sets, shadowed sets, orthopairs, squares of oppositions, and granular computing have influenced greatly the development of three-way decisions.

In this paper, we describe three-way decisions as two separated tasks of trisecting and acting. Within this trisecting-and-acting framework, we examine two previous attempts to formulate three-way decisions, i.e., probabilistic three-way classifications and evaluation-based three-way decisions, and present a third attempt. The new formulation uses more generic terms and notations and avoids limitations of the previous formulations. The new formulation enables us to identify and discuss a number of research topics of three-way decisions.

For a full understanding of three-way decisions, we must cast our study in the light of results across many fields. In the next few years, we expect to see a continued growth of research interest in three-way decisions in such a wider context.

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