Rough Set Approximations in Formal Concept Analysis and Knowledge Spaces

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Abstract. This paper proposes a generalized definition of rough set approximations, based on a subsystem of subsets of a universe. The subsystem is not assumed to be closed under set complement, union and intersection. The lower or upper approximation is no longer one set but composed of several sets. As special cases, approximations in formal concept analysis and knowledge spaces are examined. The results provide a better understanding of rough set approximations.

1 Introduction

Rough set theory [6, 7] is an extension of the set theory with two additional unary set-theoretic operators known as approximation operators. One way to define approximation operators is called the subsystem-based formulation [14, 17]. With respect to an equivalence relation on a finite and nonempty universe, one can construct a subsystem of the power set of the universe, which is the σ-algebra with the family of equivalence classes as a basis. The elements of the subsystem may be understood as definable or observable sets. Every subset of the universe is approximated from below and above by two sets in the subsystem.

There are two basic restrictions of the standard Pawlak model. First, a single subsystem of the power set is used. Second, the σ-algebra is closed under set complement, intersection and union. Many studies on generalized rough set approximations try to remove those restrictions. For example, in the abstract approximation space [1], topological rough set models [9, 10, 12, 19, 20], and closure rough set models [14], two subsystems are used; one for lower approximation and another for upper approximation. In the context of formal concept analysis, one considers a subsystem that is only closed under set intersection [18]. In this paper, we further generalize the rough set model by considering subsystems without these restrictions. The generalized approximations are applied to both formal concept analysis [5, 18] and knowledge spaces [2–4].

Formal concept analysis [5, 18] is developed based on a formal context, which is a binary relation between a set of objects and a set of attributes or properties.

From a formal context, one can construct (objects, properties) pairs known as the formal concepts [5, 11]. The set of objects is referred to as the extension, and the set of properties as the intension, of a formal concept. They uniquely determine each other. The family of all formal concepts is a complete lattice. The extension of a formal concept can be viewed as a definable set of objects, although in a sense different from that of rough set theory [15, 16]. The family of extensions of all formal concepts forms a subsystem of the power set of objects. This subsystem is closed under set intersection. Thus, one can immediately study approximation operators based on the subsystem introduced in formal concept analysis [18].

The theory of knowledge spaces [2–4] represents a new paradigm in mathematical psychology. It provides a systematic approach for knowledge assessment by considering a finite set of questions and a collection of subsets of questions called knowledge states. The family of knowledge states may be determined by the dependency of questions or the mastery of different sets of questions by a group of students. The knowledge states can be viewed as definable or observable sets. The family of knowledge states forms a subsystem of the power set of questions that is only closed under set union. Similarly, approximations can be defined based on the system of knowledge states.

The generalized subsystem-based formulation of approximation operators enables us to study approximations in two related areas of formal concept analysis and knowledge spaces. The results not only lead to more insights into rough set approximations, but also bring us closer to a common framework for studying the two related theories.

2 Subsystem-based Formulation of Pawlak Rough Set Approximations

Suppose U is a finite and nonempty universe of objects. Let $E \subseteq U \times U$ be an equivalence relation on U. The equivalence relation divides the universe into a family of pair-wise disjoint subsets, called the partition of the universe and denoted by U/E. The pair $apr = (U, E)$ is referred to as an approximation space.

An approximation space induces a granulated view of the universe. For an object $x \in U$, the equivalence class containing x is given by:

$$[x]_E = \{y \mid xEy\}. \tag{1}$$

Intuitively speaking, objects in $[x]_E$ are indistinguishable from x. Under an equivalence relation, equivalence classes are the smallest non-empty observable, measurable, or definable subsets of U. By extending the definability of equivalence classes, we assume that the union of some equivalence classes is also definable. The family of definable subsets contains the empty set $\emptyset$ and is closed under set complement, intersection, and union. It is an σ-algebra whose basis is U/E and is denoted by $\sigma(U/E) \subseteq 2^U$, where 2^U is the power set of U.

In order to explicitly expression the role of $\sigma(U/E)$, we also denote the approximation space $apr = (U, E)$ as $apr = (U, \sigma(U/E))$. A subset of objects not in $\sigma(U/E)$ is said to be undefinable. An undefinable set must be approximated from below and above by a pair of definable sets.

Definition 1 In an approximation space $apr = (U, \sigma(U/E))$, a pair of approximation operators, $\underline{apr}$, $\overline{apr} : 2^U \longrightarrow 2^U$, are defined by:

$$\begin{aligned} \underline{apr}(A) &= \cup\{X \in \sigma(U/E) \mid X \subseteq A\}, \\ \overline{apr}(A) &= \cap\{X \in \sigma(U/E) \mid A \subseteq X\}. \end{aligned} \tag{2}$$

The lower approximation $\underline{apr}(A) \in \sigma(U/E)$ is the greatest definable set contained in A, and the upper approximation $\overline{apr}(A) \in \sigma(U/E)$ is the least definable set containing A. The approximation operators have the following properties: for A, $B \subseteq U$,

$$\begin{aligned}
\text{(i).}\quad & \underline{apr}(A) = (\overline{apr}(A^c))^c, \\
& \overline{apr}(A) = (\underline{apr}(A^c))^c; \\
\text{(ii).}\quad & \underline{apr}(U) = U, \\
& \overline{apr}(\emptyset) = \emptyset; \\
\text{(iii).}\quad & \underline{apr}(\emptyset) = \emptyset, \\
& \overline{apr}(U) = U; \\
\text{(iv).}\quad & \underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B), \\
& \overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B); \\
\text{(v).}\quad & \underline{apr}(A) \subseteq A; \\
& A \subseteq \overline{apr}(A); \\
\text{(vi).}\quad & \underline{apr}(\underline{apr}(A)) = \underline{apr}(A), \\
& \overline{apr}(\overline{apr}(A)) = \overline{apr}(A); \\
\text{(vii).}\quad & \underline{apr}(\overline{apr}(A)) = \overline{apr}(A), \\
& \overline{apr}(\underline{apr}(A)) = \underline{apr}(A).
\end{aligned}$$

Property (i) states that the approximation operators are dual operators with respect to set complement ^{c}. Properties (ii) and (iii) indicate that rough set approximations of $\emptyset$ or U equal to itself. Property (iv) states that the lower approximation operator is distributive over set intersection $\cap$, and the upper approximation operator is distributive over set union $\cup$. By property (v), a set lies within its lower and upper approximations. Properties (vi) and (vii) deal with the compositions of lower and upper approximation operators. The result of the composition of a sequence of lower and upper approximation operators is the same as the application of the approximation operator closest to A.

3 Generalized Rough Set Approximations

An approximation space $apr = (U, E)$ defines uniquely a topological space $(U, \sigma(U/E))$, in which $\sigma(U/E)$ is the family of all open and closed sets [10].

Moreover, the family of open sets is the same as the family of closed sets. The lower approximation operator defined by equation (2) is well-defined as long as the subsystem is closed under union. Similarly, the upper approximation operator is well-defined as long as the subsystem is closed under intersection. One may use two subsystems [1, 13]. The subsystem for lower approximation operator must be closed under union, and the subsystem for upper approximation operator must be closed under intersection. In order to keep the duality of approximation operators, elements of two subsystems must be related to each other through set complement [13]. For further generalizations of the subsystem-based definition, we remove those restrictions.

3.1 Generalized Rough Set Approximations

The definition of generalized rough set approximations is related to the formulation of abstract approximation spaces introduced by Cattaneo [1]. We focus on set-theoretic setting and remove some axioms of an abstract approximation space.

Let $\mathcal{S}_l, \mathcal{S}_u \subseteq 2^U$ be two subsystems of 2^U. The triplet $apr = (U, \mathcal{S}_l, \mathcal{S}_u)$ is called an approximation space. We impose two conditions on $\mathcal{S}_l$ and $\mathcal{S}_u$:

$$\begin{array}{lll} \text{(a).} & \emptyset \in \mathcal{S}_l, & \emptyset \in \mathcal{S}_u; \\ \text{(b).} & U \in \mathcal{S}_l, & U \in \mathcal{S}_u. \end{array}$$

The elements of $\mathcal{S}_l$ may be understood as one family of definable or observable sets. The elements of $\mathcal{S}_u$ may be understood as another family of definable or observable sets. Our objective is to approximate an undefinable set in $2^U - \mathcal{S}_l$ from below by definable sets in $\mathcal{S}_l$ and in $2^U - \mathcal{S}_u$ from above by definable sets in $\mathcal{S}_u$.

Definition 2 In an abstract approximation space $apr = (U, \mathcal{S}_l, \mathcal{S}_u)$, the lower approximation and the upper approximation are defined by:

$$\begin{aligned} \underline{apr}(A) &= \{X \in \mathcal{S}_l \mid X \subseteq A, \forall X' \in \mathcal{S}_l (X \subset X' \Longrightarrow X' \nsubseteq A)\}, \\ \overline{apr}(A) &= \{X \in \mathcal{S}_u \mid A \subseteq X, \forall X' \in \mathcal{S}_u (X' \subset X \Longrightarrow A \nsubseteq X')\}. \end{aligned} \quad (3)$$

For simplicity, the same symbols are used for generalized approximations. The lower approximation $\underline{apr}(A)$ is the set of maximal elements of the set $\{X \in \mathcal{S}_l | X \subseteq A\}$ and the upper approximation $\overline{apr}(A)$ is the set of minimal elements of the set $\{X \in \mathcal{S}_u | A \subseteq X\}$. The definition is a generalization of Definition 1. The generalized lower and upper approximation operators have the following properties:

$$\begin{array}{ll} (1). & \underline{apr}(\emptyset) = \{\emptyset\}, \\ & \overline{apr}(\emptyset) = \{\emptyset\}; \\ (2). & \underline{apr}(U) = \{U\}, \\ & \overline{apr}(U) = \{U\}; \end{array}$$

$$\begin{aligned}
(3).\quad & A \subseteq B \Longrightarrow (\exists X \in \underline{apr}(A), \exists Y \in \underline{apr}(B), X \subseteq Y),\\
& A \subseteq B \Longrightarrow (\exists X \in \overline{apr}(A), \exists Y \in \overline{apr}(B), X \subseteq Y);\\
(4).\quad & X \in \underline{apr}(A) \Longrightarrow X \subseteq A,\\
& X \in \overline{apr}(A) \Longrightarrow A \subseteq X;\\
(5).\quad & X \in \underline{apr}(A) \Longrightarrow \underline{apr}(X) = \{X\},\\
& X \in \overline{apr}(A) \Longrightarrow \overline{apr}(X) = \{X\};\\
(6).\quad & X \in \overline{apr}(A) \Longrightarrow \underline{apr}(X) = \{X\},\\
& X \in \underline{apr}(A) \Longrightarrow \overline{apr}(X) = \{X\}.
\end{aligned}$$

They easily follow from the definition of generalized approximation operators.

3.2 Special Cases

We discuss several types of generalized rough set approximations under different conditions.

Case 1: $\mathcal{S}_l$ is closed under set union and $\mathcal{S}_u$ is closed under set intersection. If $\mathcal{S}_l$ is closed under union, the lower approximation is composed of one set defined by Definition 1. That is,

$$\underline{apr}(A) = \{\cup\{X \in \mathcal{S}_l \mid X \subseteq A\}\}. \tag{4}$$

Similarly, if $\mathcal{S}_u$ is closed under intersection, the upper approximation is composed of one set defined by Definition 1. That is,

$$\overline{apr}(A) = \{\cap\{X \in \mathcal{S}_u \mid A \subseteq X\}\}. \tag{5}$$

Case 2: $\mathcal{S}_l$ and $\mathcal{S}_u$ are dual subsystems, that is, $\mathcal{S}_u = \{X^c | X \in \mathcal{S}_l\}$ and $\mathcal{S}_l = \{X^c | X \in \mathcal{S}_u\}$. The approximations satisfy the property:

$$X \in \underline{apr}(A) \Longrightarrow X^c \in \overline{apr}(A^c). \tag{6}$$

Case 3: $\mathcal{S}_l = \mathcal{S}_u$. When $\mathcal{S}_l = \mathcal{S}_u = \mathcal{S}$, we have an approximation space $apr = (U, \mathcal{S})$. We can define the approximations as follows:

$$\begin{aligned}
\underline{apr}(A) &= \{X \in \mathcal{S} \mid X \subseteq A, \forall X' \in \mathcal{S}(X \subset X' \Longrightarrow X' \nsubseteq A)\},\\
\overline{apr}(A) &= \{X \in \mathcal{S} \mid A \subseteq X, \forall X' \in \mathcal{S}(X' \subset X \Longrightarrow A \nsubseteq X')\}.
\end{aligned} \tag{7}$$

Case 4: $\mathcal{S}_l = \mathcal{S}_u = \mathcal{S}$ and is closed under set complement. The approximations are the same as defined by equation (7). It also satisfies the property of equation (6).

Case 5: $\mathcal{S}_u = \mathcal{S}$ is closed under set intersection and $\mathcal{S}_l = \mathcal{S}^c$ is closed under set union. We define:

$$\begin{aligned}
\underline{apr}(A) &= \{\cup\{X \in \mathcal{S}^c \mid X \subseteq A\}\},\\
\overline{apr}(A) &= \{\cap\{X \in \mathcal{S} \mid A \subseteq X\}\}.
\end{aligned} \tag{8}$$

They correspond to rough set approximations in closure systems [14]. Since a closure system is only closed under set intersection, the lower and upper approximation operators satisfy less properties, as characterized by properties (iii), (v), (vi).

Case 6: $\mathcal{S}_l = \mathcal{S}$ is closed under set union and intersection and $\mathcal{S}_u = \mathcal{S}^c$ is closed under set union and intersection. We define:

$$\begin{aligned} \underline{apr}(A) &= \{\cup\{X \in \mathcal{S} \mid X \subseteq A\}\}, \\ \overline{apr}(A) &= \{\cap\{X \in \mathcal{S}^c \mid A \subseteq X\}\}. \end{aligned} \tag{9}$$

They correspond to rough set approximations in topological spaces [10]. They are in fact the topological interior and closure operators satisfy properties (i) - (vi).

Case 7: $\mathcal{S}_l = \mathcal{S}_u = \mathcal{S}$ and is closed under set complement, intersection and union. This is the standard Pawlak rough set model.

4 Approximations in Formal Concept Analysis

Let U and V be any two finite sets. Elements of U are called objects, and elements of V are called properties. The relationships between objects and properties are described by a binary relation R between U and V, which is a subset of the Cartesian product $U \times V$. For a pair of elements $x \in U$ and $y \in V$, if $(x, y) \in R$, written as xRy, x has the property y, or the property y is possessed by object x. The triplet (U, V, R) is called a formal context.

Based on the binary relation, we associate a set of properties to an object. An object $x \in U$ has the set of properties:

$$xR = \{y \in V \mid xRy\} \subseteq V. \tag{10}$$

Similarly, a property y is possessed by the set of objects:

$$Ry = \{x \in U \mid xRy\} \subseteq U. \tag{11}$$

By extending these notations, we can establish relationships between subsets of objects and subsets of properties. This leads to two operators, one from 2^U to 2^V and the other from 2^V to 2^U.

Definition 3 Suppose (U, V, R) is a formal context. For a subset of objects, we associate it with a set of properties:

$$\begin{aligned} X^* &= \{y \in V \mid \forall x \in U (x \in X \Longrightarrow xRy)\} \\ &= \{y \in V \mid X \subseteq Ry\} \\ &= \bigcap_{x \in X} xR. \end{aligned} \tag{12}$$

For a subset of properties, we associate it with a set of objects:

$$\begin{aligned} Y^* &= \{x \in U \mid \forall y \in V (y \in Y \Longrightarrow xRy)\} \\ &= \{x \in U \mid Y \subseteq xR\} \\ &= \bigcap_{y \in Y} Ry. \end{aligned} \tag{13}$$

A pair (X, Y), with $X \subseteq U$ and $Y \subseteq V$, is called a formal concept of the context (U, V, R), if $X = Y^*$ and $Y = X^*$. Furthermore, $X = ex(X, Y)$ is called the extension of the concept, and $Y = in(X, Y)$ is the intension of the concept. The set of all formal concepts forms a complete lattice called a concept lattice, denoted by $L(U, V, R)$ or simply L.

A formal concept consists of a definable set of objects and a definable set of properties. The concept lattice is the family of all such definable concepts. Given an arbitrary set of objects, it may not be the extension of a formal concept. The set can therefore be viewed as an undefinable set of objects. Such a set of objects can be approximated by definable sets of objects, namely, the extensions of formal concepts. Approximation operators can be introduced by using the subsystem-based formulation of rough set theory, based on the combination of case 1 and case 3.

Definition 4 For a formal concept lattice L, the family of all extensions is given by:

$$EX(L) = \{ex(X, Y) \mid (X, Y) \in L\}. \tag{14}$$

The system $EX(L)$ contains the empty set $\emptyset$, the entire set U, and is closed under intersection. It defines an approximation space $apr = (U, EX(L))$. One can keep the intuitive interpretations of lower and upper approximations. That is, the lower approximation is a largest set in $EX(L)$ that is contained in A, and the upper approximation is a smallest set in $EX(L)$ that contains A. In this case, since the system is not closed under union, the smallest set containing A is unique, while the largest set contained in A is no longer unique.

Definition 5 In the approximation space $apr = (U, EX(L))$, for a subset of objects $A \subseteq U$, its upper approximation is defined by:

$$\overline{apr}(A) = \{\cap\{X \in EX(L) \mid A \subseteq X\}\}, \tag{15}$$

and its lower approximation is a family of sets:

$$\underline{apr}(A) = \{X \in EX(L) \mid X \subseteq A, \forall X' \in EX(L)(X \subset X' \Longrightarrow X \not\subseteq A)\}. \tag{16}$$

Thus, in formal concept analysis, a set can be approximated from below by several definable sets of objects.

5 Approximations in Knowledge Spaces

In knowledge spaces, one uses a finite set of universe (i.e., questions denoted by Q) and a collection of subsets of the universe (i.e., a knowledge structure denoted by $\mathcal{K}$), where $\mathcal{K}$ contains at least the empty set $\emptyset$ and the whole set Q. The members of $\mathcal{K}$ are called the knowledge states which are the subsets of questions given by experts or correctly answered by students. In knowledge spaces, there are two types of knowledge structures. One is the knowledge structure associated to a surmise relation, closed under set union and intersection. Another is the knowledge structure associated to a surmise system called a knowledge space, closed under set union. The knowledge states can be viewed as a family of definable sets of objects. An arbitrary subset of questions can be approximated by knowledge states in each of the two structures. Approximation operators are introduced by using the subsystem-based formulation of rough set theory.

In knowledge spaces, a surmise relation on the set Q of questions is a transitive and reflexive relation S on Q. By aSb, we can surmise that the mastery of a if a student can answer correctly question b. This relation imposes conditions on the corresponding knowledge structure. For example, mastery question a from mastery of question b means that if a knowledge state contains b, it must also contain a.

Definition 6 For a surmise relation S on the (finite) set Q of questions, the associated knowledge structure $\mathcal{K}$ is defined by:

$$\mathcal{K} = \{K \mid (\forall q, q' \in Q, qSq', q' \in K) \Longrightarrow q \in K\}. \tag{17}$$

The knowledge structure associated to a surmise relation contains the empty set $\emptyset$, the entire set Q, and is closed under set intersection and union. It defines an approximation space $apr = (Q, \mathcal{K})$.

Definition 7 In the approximation space $apr = (Q, \mathcal{K})$, for a subset of objects $A \subseteq Q$, we define:

$$\begin{aligned} \underline{apr}(A) &= \{\cup\{K \in \mathcal{K} \mid K \subseteq A\}\}, \\ \overline{apr}(A) &= \{\cap\{K \in \mathcal{K} \mid A \subseteq K\}\}. \end{aligned} \tag{18}$$

The definition is based on the case 1. The knowledge structure associated to a surmise relation is not closed under complement, namely, it does not satisfy the duality property.

With surmise relations, a question can only have one prerequisite. This is sometimes not appropriate. In practice, we may assume that a knowledge structure is closed only under union, called a knowledge space. A knowledge space is a weakened knowledge structure associated to a surmise relation. It is a knowledge structure associated to a surmise system.

A surmise system on a (finite) set Q is a mapping σ that associates to any element q in Q a nonempty collection $\sigma(q)$ of subsets of Q satisfying the following three conditions: 1) $C \in \sigma(q) \Longrightarrow q \in C$; 2) $(C \in \sigma(q), q' \in C) \Longrightarrow (\exists C' \in \sigma(q'), C' \subseteq C)$; 3) $C \in \sigma(q) \Longrightarrow (\forall C' \in \sigma(q), C' \not\subseteq C)$. The subsets in $\sigma(q)$ are the clauses for question q.

Definition 8 For a surmise system (Q, σ), the knowledge states of the associated knowledge structure are all the subsets K of Q that satisfy:

$$\mathcal{K} = \{K \mid (\forall q \in Q, q \in K) \Longrightarrow (\exists C \in \sigma(q), C \subseteq K)\} \tag{19}$$

They constitute the knowledge structure associated to (Q, σ). It defines an approximation space $apr = (Q, \mathcal{K})$. Any knowledge structure which is closed under union is called a knowledge space. In fact, there is a one-to-one correspondence between surmise systems on Q and knowledge spaces on Q.

Compared with the system in formal concept analysis that is closed under set intersection, knowledge spaces are opposite. Being closed under set union, the lower approximation in knowledge spaces is unique while the upper approximation is a family of sets.

Definition 9 Suppose (Q, σ) is a surmise system and $\mathcal{K} \subseteq 2^Q$ is the associated knowledge structure, closed under union. In the approximation space $apr = (Q, \mathcal{K})$, for a subset of objects $A \subseteq Q$, its lower approximation is defined by:

$$\underline{apr}(A) = \{\cup\{K \in \mathcal{K} \mid K \subseteq A\}\}, \tag{20}$$

and its upper approximation is a family of sets:

$$\overline{apr}(A) = \{K \in \mathcal{K} \mid A \subseteq K, \forall K' \in \mathcal{K}(K' \subset K \Longrightarrow A \nsubseteq K'\}. \tag{21}$$

The definition is based on a combination of case 1 and case 3. The lower approximation is the largest set in $\mathcal{K}$ that contained in A, and the upper approximation is the smallest sets in $\mathcal{K}$ that contains A. While the largest set contained in A is unique, the smallest set containing A is not unique.

6 Conclusion

We propose a subsystem-based generalization of rough set approximations by using subsystems that are not closed under set complement, intersection and union. The generalized rough set approximations are not necessarily unique, but consist of a family of sets. We investigate special cases under different conditions, including subsystems that are closed under set complement, intersection and union, as well as their combinations. To show that usefulness of the proposed generalizations, approximations in formal concept analysis and knowledge spaces are examined.

The subsystem of definable sets of objects in formal concept analysis is only closed under set intersection. There are two types of subsystems in knowledge spaces. The knowledge states of a surmise relation produce a subsystem that is closed under both intersection and union. The knowledge states of a surmise system produce a subsystem that is only closed under union. The introduction of rough set approximations to the two theories demonstrates the potential value of generalized rough set approximation operators.

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