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GENERALIZATION OF ROUGH SETS USING MODAL LOGICS

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ABSTRACT—The theory of rough sets is an extension of set theory with two additional unary set-theoretic operators defined based on a binary relation on the universe. These two operators are related to the modal operators in modal logics. By exploring the relationship between rough sets and modal logics, this paper proposes and examines a number of extended rough set models. By the properties satisfied by a binary relation, such as serial, reflexive, symmetric, transitive, and Euclidean, various classes of algebraic rough set models can be derived. They correspond to different modal logic systems. With respect to graded and probabilistic modal logics, graded and probabilistic rough set models are also discussed.

***Key Words*:** rough sets, modal logic, rough set operators, graded rough sets, probabilistic rough sets

1. INTRODUCTION

The theory of rough sets is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations.[25] These approximations can be formally defined by two operators on subsets of the universe.[16] This framework provides a systematic method for the study of intelligent systems characterized by insufficient or incomplete information.[27] The successful applications of the rough set theory in a variety of problems have amply demonstrated its usefulness and versatility.[27]

A key notion in Pawlak rough set model is an equivalence relation, i.e., a reflexive, symmetric and transitive relation.[25] The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalent classes which are subsets of the set, and the upper approximation is the union of all the equivalent classes which have a non-empty intersection with the set. The requirement of an equivalent relation seems to be a stringent condition that may limit the application domain of the Pawlak rough set model. [14,15] To resolve this problem, many proposals have been made. Zakowski suggested that one may use a compatibility relation (i.e., a reflexive and symmetric relation) instead of an equivalence relation.[43] More general binary relations were used in the neighborhood systems introduced by Lin.[13] The notion of weak discernibility relation adopted by Vakarelov is a compatibility relation.[34] Since an equivalence relation is a compatibility relation, the adoption of a compatibility relation generalizes the Pawlak rough set model. Wybraniec-Skardowska introduced different rough set models based on various types of binary relations.[40] Pawlak pointed out that any type of relations may be assumed on the universe for the development of a rough set theory.[28] Wong, Wang and Yao extended further the notion of rough sets through a binary relation between elements of two universes.[38] The rough set model can be obtained in the case where the two universes become the same.

There have been extensive studies on the logical foundation of the theory of rough sets and its relationships to non-standard logics.[26,27] For example, Orlowska proposed a logic for reasoning about concepts using the notion of rough sets, which is essentially the modal logic system S5 with the modal operators interpreted using the lower and upper approximations.[23] A similar approach was also adopted by Chakraborty and Banerjee.[3] Vakarelov considered the lower and upper approximations formed from different types of relations as additional and distinct modal operators.[34] The semantics of these logic systems have been investigated by many authors.[19, 24]

Using the relationship between rough sets and modal logics, we extended conventional rough set models by considering various types of relations.[41] By generalizing our preliminary results to a wider context, this paper provides a systematic study on the generalization of rough sets using modal logics. A rough set theory is established as a non-standard set theory with two additional set-theoretic operators,[16] in the same manner that modal logic is proposed as an extension of propositional logic with two modal operators.[4, 33] These approximation operators are defined with respect to a binary relation on the universe. The properties of the binary relation determine the properties of the approximation operators. Consequently, different rough set models can be classified according to the properties of the binary relation. With respect to normal modal logics, a number of classes of algebraic rough models are analyzed and their relationships are examined. Similarly, graded and probabilistic rough set models are defined and examined based on graded and probabilistic modal logics.

The discussion of this article is essentially parallel to that in the exposition of modal logic and draws heavily from the literatures of the latter. To emphasize their similarities, the same labels are used to name the corresponding properties of the approximation operators and that of the modal operators.

2. ALGEBRAIC ROUGH SETS

In this section, we establish a link between Pawlak rough sets and modal logic S5. By applying the same relation in a wider context, we identify and classify other types of rough set models.

2.1 Pawlak Rough Sets and Modal Logic S5

Let U denote a finite and non-empty set called the universe, and let $\Re \subseteq U \times U$ denote an equivalence relation on U. The pair $apr = (U, \Re)$ is called an approximation space. The equivalence relation $\Re$ partitions the set U into disjoint subsets. Such a partition of the universe is denoted by $U / \Re$. If two elements x, y in U belong to the same equivalence class, we say that x and y are indistinguishable. The equivalence classes of $\Re$ and the empty set $\varnothing$ are called the elementary or atomic sets in the approximation space $apr = (U, \Re)$.

Given an arbitrary set $A \subseteq U$, it may be impossible to describe A precisely using the equivalence classes of $\Re$. In this case, one may characterize A by a pair of lower and upper approximations:

$$\begin{aligned} \underline{apr}(A) &= \bigcup_{[x]_\Re \subseteq A} [x]_\Re \\ \overline{apr}(A) &= \bigcup_{[x]_\Re \cap A \neq \varnothing} [x]_\Re \end{aligned} \tag{1}$$

where

$$[x]_\Re = \{y \mid x \Re y\} \tag{2}$$

is the equivalence class containing x. The pair $(\underline{apr}(A), \overline{apr}(A))$ is called the rough set with respect to A. The lower approximation $\underline{apr}(A)$ is the union of all the elementary sets which are subsets of A, and the upper approximation $\overline{apr}(A)$ is the union of all the elementary sets which have a non-empty intersection with A. An element in the lower approximation necessarily belongs to A, while an element in the upper approximation possibly belongs to A. Equivalently, we can reexpress lower and upper approximations as follows:

$$\begin{aligned} \underline{apr}(A) &= \{x \mid [x]_{\Re} \subseteq A\} \\ &= \{x \in U \mid \text{for all } y \in U,\ x\Re y \text{ implies } y \in A\} \\ \overline{apr}(A) &= \{x \mid [x]_{\Re} \cap A \neq \varnothing\} \\ &= \{x \in U \mid \text{there exists a } y \in U \text{ such that } x\Re y \text{ and } y \in A\} \end{aligned} \tag{3}$$

That is, an element of U necessarily belongs to A if all its equivalent elements belong to A; it possibly belongs to A if at least one of its equivalent elements belongs to A. Such an interpretation is closely related to the necessity and possibility operators in modal logic.

For any subsets $A, B \subseteq U$, the lower approximation $\underline{apr}$ satisfies properties:

(AL1) $\underline{apr}(A) = \sim \overline{apr}(\sim A)$

(AL2) $\underline{apr}(U) = U$

(AL3) $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$

(AL4) $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$

(AL5) $A \subseteq B \Rightarrow \underline{apr}(A) \subseteq \underline{apr}(B)$

(AL6) $\underline{apr}(\varnothing) = \varnothing$

(AL7) $\underline{apr}(A) \subseteq A$

(AL8) $A \subseteq \underline{apr}(\overline{apr}(A))$

(AL9) $\underline{apr}(A) \subseteq \underline{apr}(\underline{apr}(A))$

(AL10) $\overline{apr}(A) \subseteq \underline{apr}(\overline{apr}(A))$

and the upper approximation $\overline{apr}$ satisfies properties:

(AU1) $\overline{apr}(A) = \sim \underline{apr}(\sim A)$

(AU2) $\overline{apr}(\varnothing) = \varnothing$

(AU3) $\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$

(AU4) $\overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B)$

(AU5) $A \subseteq B \Rightarrow \overline{apr}(A) \subseteq \overline{apr}(B)$

(AU6) $\overline{apr}(U) = U$

(AU7) $A \subseteq \overline{apr}(A)$

(AU8) $\overline{apr}(\underline{apr}(A)) \subseteq A$

(AU9) $\overline{apr}(\overline{apr}(A)) \subseteq \overline{apr}(A)$

(AU10) $\overline{apr}(\underline{apr}(A)) \subseteq \underline{apr}(A)$

where $\sim A = U - A$ denotes the set complement of A. Moreover, lower and upper approximations obey properties:

(K) $\underline{apr}(\sim A \cup B) \subseteq \sim \underline{apr}(A) \cup \underline{apr}(B)$

(ALU) $\underline{apr}(A) \subseteq \overline{apr}(A)$

Properties (AL1) and (AU1) state that two approximation operators are dual operators. Hence, properties with the same number may be regarded as dual properties. These properties are not independent. For example, property (AL3) implies property (AL4). Properties (AL9), (AL10), (AU9) and (AU10) are expressed in terms of set inclusion. The standard version using set equality can be derived from (AL1)–(AL10) and (AU1)–(AU10). For example, it follows from (AL7) and (AL9) that $\underline{apr}(A) = \underline{apr}(\underline{apr}(A))$.

In this study, rough sets described above are called Pawlak rough sets. They are constructed from an equivalence relation. The pair of lower and upper approximations may be interpreted as two operators $\underline{apr}$ and $\overline{apr}$ on subsets of U.[16] Under this view, the rough set theory may be regarded an extension of set theory with two additional unary set-theoretic operators. Together with the standard set-theoretic operators, they define a mathematical system $(2^U, \cap, \cup, \sim, \underline{apr}, \overline{apr})$. It is an extension of the Boolean algebra $(2^U, \cap, \cup, \sim)$ and is referred to as Pawlak rough set model. From the properties satisfied by lower and upper approximation operators, Pawlak rough set model may be interpreted in terms of the notions of topological space and topological Boolean algebra.[25, 32, 37] In this system, the proposed operators $\underline{apr}$ and $\overline{apr}$ can be used together with the usual set-theoretic operators $\sim$, $\cap$, and $\cup$ to form valid expressions regarding sets. For example, $\sim \underline{apr}(\overline{apr}(A) \cup B)$ is a valid expression.

The modal logic S5 may be understood algebraically by topological Boolean algebra.[12, 32] It is natural to expect that there is connection between Pawlak rough set model and modal logic S5. Such a link can be formally established below.[3, 19, 23, 24, 26, 27, 34]

Let Φ be a non-empty set of propositions, which is generated by a finite set of logical connectives, $\wedge, \vee, \ldots$, etc., propositional constants $\top$ and $\perp$, and infinitely enumerable set $P = \{\phi, \psi, \ldots\}$ of propositional variables. That is, Φ is the smallest set containing all proposition variables in P, constants $\top$ and $\perp$, and is closed under negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), and implication ($\rightarrow$). Let W be a non-empty set of possible worlds and $\mathfrak{R}$ a binary relation called an accessibility relation on W. For two possible worlds $w, w' \in W$, if $w\mathfrak{R}w'$, we say that world w' is accessible from world w. The pair $(W, \mathfrak{R})$ is referred to as a frame. An interpretation in $(W, \mathfrak{R})$ is a function υ: $W \times P \rightarrow \{true, false\}$, which assigns a truth value for each proposition variable with respect to each particular world w. If $\upsilon(w, a) = true$, we say that the proposition a is true in the interpretation υ in the world w, written $w \models_{\upsilon} a$. We extend υ to the set of all propositions Φ in a standard way, i.e., we define $\upsilon^*: W \times \Phi \rightarrow \{true, false\}$ on a degree of proposition $\phi \in \Phi$ as follows:

(m0) for $a \in P$, $w \models_{\upsilon*} a$ iff $w \models_{\upsilon} a$

(m1) not $w \models_{\upsilon*} \bot$, $w \models_{\upsilon*} \top$

(m2) $w \models_{\upsilon*} (\phi \wedge \psi)$ iff $w \models_{\upsilon*} \phi$ and $w \models_{\upsilon*} \psi$

(m3) $w \models_{\upsilon*} (\phi \vee \psi)$ iff $w \models_{\upsilon*} \phi$ or $w \models_{\upsilon*} \psi$, or both

(m4) $w \models_{\upsilon*} (\phi \rightarrow \psi)$ iff not $w \models_{\upsilon*} \phi$ or $w \models_{\upsilon*} \psi$, or both

(m5) $w \models_{\upsilon*} \neg \phi$ iff not $w \models_{\upsilon*} \phi$

For simplicity, we will write $w \models_{\upsilon} \phi$, and when υ is clear from context, we drop it by simply writing $w \models \phi$.

In addition to the standard logic connectives, modal logic introduces a necessity operator $\Box$ and a possibility operator $\Diamond$. That is, a modal logic system is an extended system $(\Phi, \wedge, \vee, \neg, \Box, \Diamond)$ of the propositional logic system $(\Phi, \wedge, \vee, \neg)$. The semantics of modal logic is defined in terms of possible worlds as follows: for $w \in W$ and $\phi, \psi \in \Phi$,

(m6) $w \models \Box\phi$ iff for all $w' \in W$, $w\Re w'$ implies $w' \models \phi$

(m7) $w \models \Diamond\phi$ iff there exists a $w' \in W$ such that $w\Re w'$ and $w' \models \phi$

The standard logic connectives have the same interpretation in both propositional logic and modal logic. A proposition ϕ is necessarily true in a possible world w, i.e., $w \models \Box\phi$, if ϕ is true in every world accessible from w; ϕ is possibly true, i.e., $w \models \Diamond\phi$, if ϕ is true in at least one world accessible from w. The necessity and possibility operators are dual operators. Each can be defined in terms of the other by:

$$\begin{aligned} \Box\phi &= \neg \Diamond \neg \phi \\ \Diamond\phi &= \neg \Box \neg \phi \end{aligned} \tag{4}$$

If the relation $\Re$ is an equivalence relation, the corresponding modal logic system is commonly known as S5.

With a valuation function υ, we can characterize a proposition by the set of possible worlds in which the proposition is true. In other words, we can define a mapping $t : \Phi \rightarrow 2^W$ as follows:

$$t(\phi) = \{w \in W \mid w \models \phi\} \tag{5}$$

The set $t(\phi)$ is referred to as the truth set of the proposition.[4] It is also called the incidence of ϕ and the mapping t is called an incidence mapping.[1] Using the truth set representation, the logical connectives can be interpreted using the set-theoretic operations. The following equations explicitly express the relationships between logic connectives and set-theoretic operators:

(s1) $t(\bot) = \varnothing$, $t(\top) = W$

(s2) $t(\phi \wedge \psi) = t(\phi) \cap t(\psi)$

(s3) $t(\phi \vee \psi) = t(\phi) \cup t(\psi)$

(s4) $t(\phi \rightarrow \psi) = {\sim}t(\phi) \cup t(\psi)$

(s5) $t(\neg \phi) = {\sim}t(\phi)$

$$\text{(s6)} \qquad t(\Box\phi) = \underline{apr}(t(\phi))$$
$$\text{(s7)} \qquad t(\Diamond\phi) = \overline{apr}(t(\phi))$$

The first five properties are straightforward. For example, (s2) follows from that fact that proposition $\phi \wedge \psi$ is true in a world w if and only if both ϕ and ψ are true in w.

The last two properties can be derived from equation (3):

$$\begin{aligned}
t(\Box\phi) &= \{w \in W \mid w \models \Box\phi\} \\
&= \{w \in W \mid \text{for all } w \in W,\ w\Re w' \text{ implies } w' \models \phi\} \\
&= \{w \in W \mid \text{for all } w \in W,\ w\Re w' \text{ implies } w' \in t(\phi)\} \\
&= \underline{apr}(t(\phi)) \\
t(\Diamond\phi) &= \{w \in W \mid w \models \Diamond\phi\} \\
&= \{w \in W \mid \text{there exists a } w' \in W \text{ such that } w\Re w' \text{ and } w' \models \phi\} \\
&= \{w \in W \mid \text{there exists a } w' \in W \text{ such that } w\Re w' \text{ and } w' \in t(\phi)\} \\
&= \overline{apr}(t(\phi))
\end{aligned}$$

Such an interpretation was also used by many authors (e.g., Chakraborty and Banerjee,[3] Orlowska,[23] and Pawlak[26]).

By the above interpretation, the properties of approximation operators in Pawlak rough set model are related to the properties of modal operators in molal logic S5. Each of the properties of Pawlak rough sets becomes a property of modal logic, if $\underline{apr}$ is replaced by $\Box$, $\overline{apr}$ by $\Diamond$, $\cap$ by $\wedge$, $\cup$ by $\vee$, $\sim$ by $\neg$, $=$ by $\leftrightarrow$, and subsets of U by propositions of Φ. For example, property (AL3) corresponds to $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$. By adopting the labeling system used by Chellas,[4] axioms of modal logic corresponding to (AK), (ALU), (AL7)–(AL10) are given by:

$$\begin{aligned}
&\text{(K)} && \Box(\phi \to \psi) \to (\Box\phi \to \Box\phi) \\
&\text{(D)} && \Box\phi \to \Diamond\phi \\
&\text{(T)} && \Box\phi \to \phi \\
&\text{(B)} && \phi \to \Box\Diamond\phi \\
&\text{(4)} && \Box\phi \to \Box\Box\phi \\
&\text{(5)} && \Diamond\phi \to \Box\Diamond\phi
\end{aligned}$$

The established link shows that Pawlak rough set model is a counterpart of modal logic S5. The standard logic operators are interpreted by usual set-theoretic operators, and the modal operators by rough set operators.

In modal logic, different systems can be constructed by using various types of binary relations. A number of important questions immediately arise. Is it possible to construct different rough set models with respect to various modal logic systems? If the answer is positive, how to construct these rough set model? The rest of this section will address these issues.

2.2 Generalized Rough Set Operators

The Pawlak rough set model may be extended by using an arbitrary binary relation in the same way modal operators are defined. In this framework, the generalized rough set operators are still related to modal operators as shown by properties (s6) and (s7). An advantage of such a formulation is that results from modal logic can be immediately applied.

Given a binary relation $\Re$ and two elements x, $y \in U$, if $x\Re y$, we say that y is $\Re$-related to x. A binary relation may be more conveniently represented by a mapping $r : U \rightarrow 2^U$:

$$r(x) = \{y \in U | x\Re y\} \tag{6}$$

That is, $r(x)$ consists of all $\Re$-related elements of x. By extending Equation (3), we define two unary set-theoretic operators $\underline{apr}$ and $\overline{apr}$:

$$\begin{aligned} \underline{apr}(A) &= \{x | r(x) \subseteq A\} \\ &= \{x \in U | \text{for all } y \in U,\ x\Re y \text{ implies } y \in A\} \\ \overline{apr}(A) &= \{x | r(x) \cap A \neq \varnothing\} \\ &= \{x \in U | \text{there exists a } y \in U \text{ such that } x\Re y \text{ and } y \in A\} \end{aligned} \tag{7}$$

The set $\underline{apr}(A)$ consists of those elements whose $\Re$-related elements are all in A, and $\overline{apr}(A)$ consists of those elements such that at least one of whose $\Re$-related elements is in A. The pair $(\underline{apr}(A), \overline{apr}(A))$ is referred to as the generalized rough set of A induced by $\Re$. Its physical meaning depends on the interpretations of the universe and the relation $\Re$ in particular applications. Operators $\underline{apr}, \overline{apr}: 2^U \rightarrow 2^U$ are referred to as the generalized rough set operators. The induced system $(2^U, \cap, \cup, \sim, \underline{apr}, \overline{apr})$ is called an algebraic rough set model.

The set $r(x)$ may be interpreted as a neighborhood of x.[13,16] Hence, $\underline{apr}$ and $\overline{apr}$ are indeed the interior and closure of A. This formulation is only a special case of neighborhood systems. In neighborhood systems, one may consider several (finite or infinite) binary relations at the same time. For example, if U is a non-finite topological spaces (a special type of neighborhood systems), each element has a neighborhood systems which often consists of infinitely many neighborhoods.

Example 1 Let $U = \{a, b, c\}$. Consider a binary relation $\Re$ defined by:

$$a\Re a,\quad b\Re b,\quad a\Re b,\quad b\Re a,\quad c\Re b.$$

From Equation (6), $\Re$-related elements for each member of U are given by:

$$r(a) = \{a, b\},\quad r(b) = \{a, b\},\quad r(c) = \{b\}.$$

According to Equation (7), the two extended operators are defined as:

$$\begin{aligned} &\underline{apr}(\varnothing) = \varnothing, && \overline{apr}(\varnothing) = \varnothing, \\ &\underline{apr}(\{a\}) = \varnothing, && \overline{apr}(\{a\}) = \{a,b\}, \\ &\underline{apr}(\{b\}) = \{c\}, && \overline{apr}(\{b\}) = \{a,b,c\}, \\ &\underline{apr}(\{c\}) = \varnothing, && \overline{apr}(\{c\}) = \varnothing, \end{aligned}$$

$$\underline{apr}(\{a,b\}) = \{a,b,c\}, \qquad \overline{apr}(\{a,b\}) = \{a,b,c\},$$

$$\underline{apr}(\{a,c\}) = \varnothing, \qquad \overline{apr}(\{a,c\}) = \{a,b\},$$

$$\underline{apr}(\{b,c\}) = \{c\}, \qquad \overline{apr}(\{b,c\}) = \{a,b,c\},$$

$$\underline{apr}(U) = U, \qquad \overline{apr}(U) = U,$$

The definition of generalized rough set operators is obviously consistent with the interpretation of necessity and possibility operators in modal logic. More specifically, (s6) and (s7) hold for generalized rough set operators:

$$\text{(s6)} \quad t(\Box\phi) = \underline{apr}(t(\phi)),$$

$$\text{(s7)} \quad t(\Diamond\phi) = \overline{apr}(t(\phi)).$$

In the case where $\mathfrak{R}$ is an equivalence relation, generalized rough set operators reduce to the operators in Pawlak rough set model. If $\mathfrak{R}$ is a compatibility relation, generalized rough set operators are different from the ones proposed by Pomykala[31] and Zakowski.[43] They used a covering of the universe to define rough sets instead of using a relation explicitly. In general, the generalized rough set operators are consistent with proposals by Kortelainen,[12] Lin and Liu,[16] Wong, Wang and Yao,[38] and Wybraniec-Skardowska.[40]

2.3 Classification of Algebraic Rough Set Models

For an arbitrary relation, generalized rough set operators do not necessarily satisfy all the properties in Pawlak rough set models. Nevertheless, properties (AL1)–(AL5) and (AU1)–(AU5) hold in any rough set model, i.e., independent of the properties of the binary relation. This can be easily seen from the definition. For instance, property (AL3) can be shown as follows:

$$\begin{aligned} x \in \underline{apr}(A \cap B) &\Leftrightarrow r(x) \subseteq A \cap B \\ &\Leftrightarrow r(x) \subseteq A \text{ and } r(x) \subseteq A \\ &\Leftrightarrow x \in \underline{apr}(A) \text{ and } x \in \underline{apr}(B) \\ &\Leftrightarrow x \in \underline{apr}(A) \cap \underline{apr}(B) \end{aligned} \tag{8}$$

That is, $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$. Property (AK) also holds in any rough set model. Suppose $x \in \underline{apr}(\sim A \cup B)$. This means that $r(x) \subseteq \sim A \cup B$. Thus, only one the following two conditions can be true: (i) $r(x) \subseteq B$ and (ii) $r(x) \not\subseteq B$ and $r(x) \cap \sim A \neq \varnothing$. For the case (i), it is obvious that $x \in \sim \underline{apr}(A) \cup \underline{apr}(B)$. For the case (ii), from $r(x) \cap \sim A \neq \varnothing$, one can conclude that $r(x) \not\subseteq A$. It implies that $x \notin \underline{apr}(A)$ and in turn $x \in \sim\underline{apr}(A)$. Therefore, $x \in \sim \underline{apr}(A) \cup \underline{apr}(B)$. By summarizing the results of both cases (i) and (ii), one can say that property (AK) holds. This property is a basic property in our formulation of rough set models.

In modal logic, properties corresponding to (AK) and (AL6)–(AL10) are used to define different logic systems. By adopting this technique, we use these properties to classify various rough set models. To be consistent with notation used in modal logic, we relabel some of these properties as follows:

(K)	$\underline{apr}(\sim A \cup B) \subseteq \sim \underline{apr}(A) \cup \underline{apr}(B)$
(D)	$\underline{apr}(A) \subseteq \overline{apr}(A)$
(T)	$\underline{apr}(A) \subseteq A$
(B)	$A \subseteq \underline{apr}(\overline{apr}(A))$
(4)	$\underline{apr}(A) \subseteq \underline{apr}(\underline{apr}(A))$
(5)	$\overline{apr}(A) \subseteq \underline{apr}(\overline{apr}(A))$

In order to construct a rough set model so that the above properties hold, it is necessary to impose certain conditions on the binary relation $\Re$. In fact, each of these properties corresponds to a property of the binary relation.

A relation $\Re$ is a serial relation if for all $x \in U$ there exists a $y \in U$ such that $x\Re y$. A relation is a reflexive relation if for all $x \in U$ the relationship $x\Re x$ holds. A relation is symmetric relation if for all $x,y \in U$, $x\Re y$ implies $y\Re x$ holds. A relation is a transitive relation if for three elements x, y, $z \in U$, $x\Re y$ and $y\Re z$ imply $x\Re z$. A relation is Euclidean when for all $x, y, z \in U$, $x\Re y$ and $x\Re z$ imply $y\Re z$. Since the approximation operators are defined through the mapping r, it is more convenient to express equivalently the conditions on a binary relation as follows:[4]

serial:	for all $x \in U$, $r(x) \neq \varnothing$
reflexive:	for all $x \in U$, $x \in r(x)$
symmetric:	for all $x,y \in U$, if $x \in r(y)$, then $y \in r(x)$
transitive:	for all $x,y \in U$, if $y \in r(x)$, then $r(y) \subseteq r(x)$
Euclidean:	for all $x,y \in U$, if $y \in r(x)$, then $r(x) \subseteq r(y)$

We name the rough set model according to the properties of the binary relation. For example, a rough set model constructed from a symmetric relation is referred to as a symmetric rough set model.

In a serial rough set model, for any $x \in \underline{apr}(A)$, we have $r(x) \subseteq A$ and $r(x) \neq \varnothing$, which imply $r(x) \cap A \neq \varnothing$, namely, $x \in \overline{apr}(A)$. Thus, property (D) holds in a serial rough set model. Based on this property, we may call $\underline{apr}(A)$ a lower representation and $\overline{apr}(A)$ an upper representation. By combining (D) with (AL2) and (AU2), it follows that $\underline{apr}(\varnothing) = \varnothing$ and $\overline{apr}(U) = U$. Therefore, a serial rough set model is indeed related to the notion of interval structures introduced by Wong, Wang and Yao,[38] in which a single universe is used. In a reflexive rough set model, for any $x \in U$, $x\Re x$ implies that $x \in r(x)$. Suppose $x \in \underline{apr}(A)$, which is equivalent $r(x) \subseteq A$. Combining $x \in r(x)$ and $r(x) \subseteq A$, we have $x \in A$. Thus, property (T) holds. In this case, a set A lies between its lower and upper representations. Consider a symmetric rough set model. Suppose $x \in A$. By the symmetry of $\Re$, for all $y \in r(x)$ we have $x \in r(y)$, i.e., $x \in r(y) \cap A$. This implies that for all $y \in r(x)$, $y \in \overline{apr}(A)$. Hence, $r(x) \subseteq \overline{apr}(A)$. It means that $x \in \underline{apr}(\overline{apr}(A))$. Therefore, property (B) holds. That is, a set A is a subset of the lower representation of its upper representation. In a transitive rough set model, suppose $x \in \underline{apr}(A)$, i.e., $r(x) \subseteq A$. Then for all $y \in r(x)$, $r(y) \subseteq r(x) \subseteq A$. This is equivalent to say that for all $y \in r(x)$, $y \in \underline{apr}(A)$. One can therefor conclude that $r(x) \subseteq \underline{apr}(A)$ and in turn $x \in \underline{apr}(\underline{apr}(A))$. That is, the property (4) holds, namely, the lower representation of a set is a subset of the lower representation of the lower representation of that set. Consider now an Euclidean rough set model. Suppose $x \in \overline{apr}(A)$, i.e., $r(x) \cap A \neq \varnothing$. By the Euclidean property of $\Re$, for all $y \in r(x)$, $r(x) \subseteq r(y)$. Combining this result with the assumption $r(x) \cap A \neq \varnothing$, we can conclude that for all $y \in r(x)$, $y \in \overline{apr}(A)$. This is equivalent to say $r(x) \subseteq \overline{apr}(A)$. Therefore, property (5) holds in an Euclidean rough set model. In this model, the upper representation of a set is a subset of the lower representation of the upper representation of that set.

The five properties of a binary relation, namely, the serial, reflexive, symmetric, transitive, and Euclidean properties, induce five properties of the approximation operators, namely,

serial:	property (D) holds,
reflexive:	property (T) holds,
symmetric:	property (B) holds,
transitive:	property (4) holds,
Euclidean:	property (5) holds.

By combining these properties, one can construct more rough set models. For instance, if $\Re$ is reflexive and symmetric, i.e., $\Re$ is a compatiblity relation, we obtain the rough set model built using a compatibility relation. A compatibility relation is a serial relation but not necessarily a transitive or an Euclidean relation. In such a model, properties (D), (T) and (B) hold and properties (4) and (5) do not hold. Following the convention of modal logic,[4] this type of rough set model is labeled by KTB, which is the set of properties satisfied by operators $\underline{apr}$ and $\overline{apr}$. The property (D) does not explicitly appear in the label because it can be obtained from (T). If $\Re$ is reflexive, symmetric, and transitive, i.e., $\Re$ is an equivalence relations, we obtain the Pawlak rough set model. An equivalence relation is both a compatibility relation and an Euclidean relation. Thus, the approximation operators satisfies all properties (D)–(5). This type of rough set model is denoted by KT5, the corresponding modal logic system is also called S5.

In general, any subset of the five properties may provide a class of rough set model. Since these properties are not an independent set, there are less number of rough set models than the number of subsets of properties. For example, the standard rough set model proposed by Pawlak can be characterized either by the subset {reflexive, symmetric, transitive} or the subset {reflexive, Euclidean}. By the results in modal logic, it is possible to construct fifteen distinct classes of rough set models.[4,18] Figure 1, adopted from Chellas[4] and Marchal,[18] summarizes the relationships between these models. A line connecting two models indicates the model in the upper level is a model in the lower level. For example, a KT5 model is a KT4 model. These lines that can be derived from the transitivity are not explicitly shown. The model K may be considered as the basic model because it does not require any special property on the binary relation. All other models are built on top of the model K, and hence it can be regarded as the weakest model. The model KT5, i.e., the Pawlak rough set model, is the strongest model. Both the interval structure model KD and the rough set model KTB lie between K and KT5.

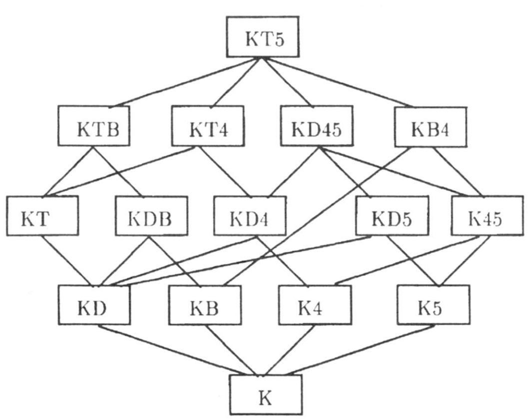

Figure 1. Rough set models.

3. GRADED AND PROBABILISTIC ROUGH SETS

In algebraic models, rough set operators are defined by using only qualitative relationships between the $\Re$-related elements of x and a subset A of U. That is, x belongs to $\underline{apr}(A)$ if all elements of $r(x)$ are in A, and x belongs to $\overline{apr}$(A) if one element of $r(x)$ is in A. The quantitative information about the degree of overlap of $r(x)$ and A is not taken into consideration. The same can be said about modal logics using only the pair of necessity and possibility operators. The number of worlds accessible from a world w, and in which a proposition is true, is not taken into consideration. It is therefore not surprising that similar efforts have been attempted in both rough sets and modal logics to incorporate such information.[5,6,7,8,9,10,42,44] The results of these studies lead to graded and probabilistic interpretations of modal logics and rough sets.

3.1 Graded Rough Sets

Graded modal logics extend modal logic by introducing a family of graded modal operators $\Box_n$ and $\Diamond_n$, where $n \in \mathbf{N}$ and $\mathbf{N}$ is the set of natural numbers.[5,7,8,35,36] These operators can be interpreted, in the usual Kripkean semantics, as follows:

$$\text{(gm6)} \quad w \models \Box_n \phi \text{ iff } |r(w)| - |t(\phi) \cap r(w)| \leq n$$
$$\text{(gm7)} \quad w \models \Diamond_n \phi \text{ iff } |t(\phi) \cap r(w)| > n$$

where $|\cdot|$ denotes the cardinality of a set. Recall that $t(\phi)$ is the set of possible worlds in which ϕ is true, and $r(w)$ is the set of possible worlds accessible from w. It follows that ϕ is true in $|t(\phi) \cap r(w)|$ possible worlds accessible from w, and ϕ is false in $|r(w)| - |t(\phi) \cap r(w)|$ possible worlds accessible from w. Therefore, the interpretation of $\Box_n\phi$ is that ϕ is false in at most n possible worlds accessible from w. The interpretation of $\Diamond_n\phi$ is that ϕ is true in more than n possible worlds accessible from w. Obviously, graded necessity and possibility operators are dual operators:

$$\begin{aligned} \Box_n\phi &= \neg\Diamond_n \neg\phi \\ \Diamond_n\phi &= \neg\Box_n \neg\phi \end{aligned} \tag{9}$$

If $n = 0$, they reduce to normal modal operators, namely,

$$\begin{aligned} \Box\phi &= \Box_0 \phi \\ \Diamond\phi &= \Diamond_0 \phi \end{aligned} \tag{10}$$

Using graded modal operators, a new operator $\Diamond!_n$ is defined by $\Diamond!_0\phi = \Box_0 \neg \phi$ and $\Diamond!_n\phi = \Diamond_{n-1} \phi \wedge \neg \Diamond_n \phi$ for $n > 0$. The interpretation of $\Diamond!_n\phi$ is that ϕ is true in exactly n possible worlds accessible from w. With graded modal operators, one may study graded modal systems corresponding to K, KT, KT5 (S5), etc. [2,5,7,8,35,36]

The graded modal logic Gr(K) is the basic model. In this system, axiom (K) is replaced by the following three axioms:[2,5,7,8,35,36]

$$\begin{aligned} &\text{(GK1)} \quad \Box_0(\phi \rightarrow \psi) \rightarrow (\Box_n\phi \rightarrow \Box_n\psi) \\ &\text{(GK2)} \quad \Box_n\phi \rightarrow \Box_{n+1}\phi \\ &\text{(GK3)} \quad \Box_0 \neg(\phi \wedge \psi) \rightarrow ((\Diamond!_n\phi \wedge \Diamond!_m\psi) \rightarrow \Diamond!_{n+m}(\phi \vee \psi)) \end{aligned}$$

(GK1) is a kind of generalized (K) axiom, (GK2) is a way to decrease grade in the possibility operator, as it is equivalent to $\Diamond_{n+1}\phi \rightarrow \Diamond_n\phi$, and (GK3) is a way to go to higher grades. The graded modal logic Gr(S5) is obtained by adding the following graded version of axioms (T) and (5):

(GT) $\Box_0\phi \rightarrow \phi$
(G5) $\Diamond_n\phi \rightarrow \Box_0\Diamond_n\phi$

If only axioms (GK) and (GT) are used, one obtains the graded modal logic Gr(KT). All other graded modal logic systems can be similarly constructed. In parallel to graded modal logic, we introduce the notion of graded rough sets. Given the universe U and a binary relation $\Re$ on U, a family of graded rough set operators are defined as:

$$\begin{aligned} \underline{apr}_n(A) &= \{x \mid |r(x)| - |A \cap r(x)| \leq n\} \\ \overline{apr}_n(A) &= \{x \mid |A \cap r(x)| > n\} \end{aligned} \tag{11}$$

An element of U belongs to $\underline{apr}_n(A)$ if at most n of its $\Re$-related elements are not in A, and belongs to $\overline{apr}_n(A)$ if more than n of its $\Re$-related elements are in A. With this definition, we establish a link between graded modal logics and rough sets:

(gs6) $t(\Box_n\phi) = \underline{apr}_n(t(\phi))$

(gs7) $t(\Diamond_n\phi) = \overline{apr}_n(t(\phi))$

That is, graded rough set operators can be interpreted in terms of graded modal operators.

Independent of the types of binary relations, graded rough set operators obey the following properties:

(GL0) $\underline{apr}(A) = \underline{apr}_0(A)$

(GL1) $\underline{apr}_n(A) = \sim \overline{apr}_n(\sim A)$

(GL2) $\underline{apr}_n(U) = U$

(GL3) $\underline{apr}_n(A \cap B) \subseteq \underline{apr}_n(A) \cap \underline{apr}_n(B)$

(GL4) $\underline{apr}_n(A \cup B) \supseteq \underline{apr}_n(A) \cup \underline{apr}_n(B)$

(GL5) $A \subseteq B \Rightarrow \underline{apr}_n(A) \subseteq \underline{apr}_n(B)$

(GL6) $n \geq m \Rightarrow \underline{apr}_n(A) \supseteq \underline{apr}_m(A)$

(GU0) $\overline{apr}(A) = \overline{apr}_0(A)$

(GU1) $\overline{apr}_n(A) = \sim \underline{apr}_n(\sim A)$

(GU2) $\overline{apr}_n(\varnothing) = \varnothing$

(GU3) $\overline{apr}_n(A \cup B) \supseteq \overline{apr}_n(A) \cup \overline{apr}_n(B)$

(GU4) $\overline{apr}_n(A \cap B) \subseteq \overline{apr}_n(A) \cap \overline{apr}_n(B)$

(GU5) $A \subseteq B \Rightarrow \overline{apr}_n(A) \subseteq \overline{apr}_n(B)$

(GU6) $n \geq m \Rightarrow \overline{apr}_n(A) \subseteq \overline{apr}_m(A)$

Properties (GL0) and (GU0) show the relationship between graded rough set operators and normal rough set operators. Properties (GL1)–(GL5) and (GU1)–(GU5) correspond to properties (AL1)–(AL5) and (AU1)–(AU5) of algebraic rough sets. For properties (GL3) and (GU3), set equality is replaced by set inclusion. Properties (GL6) and (GU6) are introduced to characterize the relationships between graded modal operators. In fact, property (GL6) corresponds to a generalized version of (GK2) of graded modal logic. Properties corresponding to (GK1) and (GK3) can be easily constructed.

Depending on the properties satisfied by the binary relation, different graded rough set models can be constructed in a similar way as discussed in the last section. If the binary relation $\Re$ is indeed an equivalence relation, we obtain the graded version of Pawlak rough sets. The operators in graded Pawlak rough sets satisfy properties corresponding to axioms of graded modal logic:

(GD) $\underline{apr}_0(A) \subseteq \overline{apr}_0(A)$

(GT) $\overline{apr}_0(A) \subseteq A$

(GB) $A \subseteq \underline{apr}_0(\overline{apr}_0(A))$

(G4) $\underline{apr}_n(A) \subseteq \underline{apr}_0(\underline{apr}_n(A))$

(G5) $\overline{apr}_n(A) \subseteq \underline{apr}_0(\overline{apr}_n(A))$

It should be noted that additional axioms may be necessary to describe other classes of graded modal logic systems. Further information about graded modal logic systems and the canonical models for them can be found in the literature.[2, 5, 7, 8, 35, 36]

3.2 *Probabilistic Rough Sets*

In the definition of graded modal operators, we only use the *absolute* number of possible worlds accessible from a world w and in which a proposition ϕ is true (false). The size of $r(w)$ is not taken into consideration. By introducing the notion of probabilistic modal logic, all such information will be used.

Suppose $(W, \Re)$ is a frame. For each $w \in W$, we define a probability function Pw: $\Phi \rightarrow [0; 1]$:

$$P_w(\phi) = \frac{|t(\phi) \cap r(w)|}{|r(w)|} \tag{12}$$

where $t(\phi)$ is the set of possible worlds in which ϕ is true, and $r(w)$ is the set of possible worlds accessible from w. In this definition, we have implicitly assumed that the binary relation $\Re$ is at least serial, i.e., for all $w \in W$, $|r(w)| \geq 1$. Using these probabilities, we define a family of probabilistic modal logic operators for $\alpha \in [0, 1]$:

(pm6) $w \models \Box_\alpha \phi$ iff $P_w(\phi) \geq 1 - \alpha$

iff $\dfrac{|t(\phi) \cap r(w)|}{|r(w)|} \geq 1 - \alpha$

(pm7) $w \models \Diamond_\alpha \phi$ iff $P_w(\phi) > \alpha$

iff $\dfrac{|t(\phi) \cap r(w)|}{|r(w)|} > \alpha$

They are related to each other by:

$$\Box_\alpha \phi = \neg \Diamond_\alpha \neg \phi$$

$$\Diamond_\alpha \phi = \neg \Box_\alpha \neg \phi \tag{13}$$

When $\alpha = 0$, they reduce to normal modal operators:

$$\Box \phi = \Box_0 \phi$$

$$\Diamond \phi = \Diamond_0 \phi \tag{14}$$

The definition of probabilistic modal operators are consistent with the proposal of Murai, Miyakoshi, and Shimbo.[17] In our formulation, probability functions are defined using the ratio of the cardinalities of $t(\phi)$ n $r(w)$ and $r(w)$, as suggested by Hart.[11] This is only a special case of the probabilistic Kripkean model proposed by Fattorosi-Barnaba and Amati.[6]

The probabilistic modal operators are related to the graded modal operators. If both sides of inequalities in (gm6) and (gm7) are divided by $|r(w)|$, and $n / |r(w)|$ is replaced by α, the probabilistic modal operators are obtained. That is, graded and probabilistic modal operators are consistently defined. However, these operators are different from each other. Consider two possible worlds $w, w' \in W$ with $|r(w) \cap t(\phi)| = |r(w') \cap t(\phi)| = 1$ and $|r(w)| \neq |r(w')|$. We have:

$$w \models \Diamond_0 \phi$$

$$w' \models \Diamond_0 \phi$$

and

$$w \models \neg \Diamond_n \phi$$

$$w' \models \neg \Diamond_n \phi$$

for $n \geq 1$. That is, evaluations of $\Diamond_n \phi$ are the same in both worlds w and w'. The difference in the sizes of $r(w)$ and $r(w')$ is reflected by operators $\Box_n$. On the other hand, since $1/|r(x)| \neq 1/|r(y)|$, evaluations of both $\Diamond_\alpha \phi$ and $\Box_\alpha \phi$ will be different in worlds w and w'. Similarly, examples can be found such that evaluations of probabilistic modal operators are the same in two worlds, while evaluations of graded modal operators are different.

We now turn attention to the definition of probabilistic rough sets. With respect to the universe U and a binary relation $\Re$ on U, we define a family of probabilistic rough set operators:

$$\underline{apr}_\alpha(A) = \left\{ x \,\middle|\, \frac{|A \cap r(x)|}{|r(x)|} \geq 1 - \alpha \right\}$$

$$\overline{apr}_\alpha(A) = \left\{ x \,\middle|\, \frac{|A \cap r(x)|}{|r(x)|} > \alpha \right\} \tag{15}$$

With this definition, the connections between probabilistic modal logic and probabilistic rough sets are established:

(ps6) $t(\Box_\alpha \phi) = \underline{apr}_\alpha(t(\phi))$

(ps7) $t(\Diamond_\alpha \phi) = \overline{apr}_\alpha(t(\phi))$

Therefore, different probabilistic rough set models may be identified and analyzed.

By definition, for a serial binary relation and $\alpha \in [0, 1]$, probabilistic rough set operators satisfy the following properties:

$$\text{(PL0)} \quad \underline{apr}(A) = \underline{apr}_0(A)$$

$$\text{(PL1)} \quad \underline{apr}_\alpha(A) = \sim \overline{apr}_\alpha(\sim A)$$

$$\text{(PL2)} \quad \underline{apr}_\alpha(U) = U$$

$$\text{(PL3)} \quad \underline{apr}_\alpha(A \cap B) \subseteq \underline{apr}_\alpha(A) \cap \underline{apr}_\alpha(B)$$

$$\text{(PL4)} \quad \underline{apr}_\alpha(A \cup B) \supseteq \underline{apr}_\alpha(A) \cup \underline{apr}_\alpha(B)$$

$$\text{(PL5)} \quad A \subseteq B \Rightarrow \underline{apr}_\alpha(A) \subseteq \underline{apr}_\alpha(B)$$

$$\text{(PL6)} \quad \alpha \geq \beta \Rightarrow \underline{apr}_\alpha(A) \supseteq \underline{apr}_\beta(A)$$

$$\text{(PU0)} \quad \overline{apr}(A) = \overline{apr}_0(A)$$

$$\text{(PU1)} \quad \overline{apr}_\alpha(A) = \sim \underline{apr}_\alpha(\sim A)$$

$$\text{(PU2)} \quad \overline{apr}_\alpha(\varnothing) = \varnothing$$

$$\text{(PU3)} \quad \overline{apr}_\alpha(A \cup B) \supseteq \overline{apr}_\alpha(A) \cup \overline{apr}_\alpha(B)$$

$$\text{(PU4)} \quad \overline{apr}_\alpha(A \cap B) \subseteq \overline{apr}_\alpha(A) \cap \overline{apr}_\alpha(B)$$

$$\text{(PU5)} \quad A \subseteq B \Rightarrow \overline{apr}_\alpha(A) \subseteq \overline{apr}_\alpha(B)$$

$$\text{(PU6)} \quad \alpha \geq \beta \Rightarrow \overline{apr}_\alpha(A) \subseteq \overline{apr}_\beta(A)$$

They are counterparts of the properties of graded rough set operators. Moreover, for $0 \leq \alpha < 0.5$,

$$\text{(PD)} \quad \underline{apr}_\alpha(A) \subseteq \overline{apr}_\alpha(A)$$

which may be interpreted as a probabilistic version of axiom (D).

Studies on probabilistic rough sets have been focused on equivalence relation. Wong and Ziarko first introduced the notion of probabilistic rough set model using probability functions defined in this section.[39] If α is chosen to be 0.5, the probabilistic rough sets is related to the proposal of Pawlak, Wong and Ziarko.[30] A detailed analysis of probabilistic rough set operators in the framework of Bayesian decision theory can be found in Yao and Wong.[42] Pawlak and Skowron referred to the conditional probabilities as rough membership functions .[29] The variable precision rough set model proposed by Ziarko adopted a similar notion.[44] In fact, the lower and upper approximations defined by Ziarko are exactly the same as those defined in this paper. Compared with modal and graded modal logics, there is a lack of systematic study on probabilistic modal logic. It will be interest to apply the results in probabilistic rough sets to probabilistic modal logic. Further study on various classes of probabilistic modal logic, as that have been done in both normal modal logic and graded modal logic, will be fruitful. Some initial results have been reported by Fattorosi-Barnaba and Amati.[6]

4. CONCLUSION

In this article, we have proposed a framework for the generalization of rough sets using modal logics. A rough set model may be viewed as an extension of set theory having two additional unary set-theoretic operators. These operators are defined with respect to a binary relation on the universe. Three types of rough sets have been examined, the algebraic rough sets, graded rough sets, and probabilistic rough sets. Within this framework, we have examined various rough set models using properties of binary relations. The algebraic rough set operators correspond to the modal operators. The graded and probabilistic rough set operators correspond to graded and probabilistic modal operators. Families of rough set models corresponds to families of modal logic systems.

The established connections between rough sets and modal logics have very important implications. Based on such relationships, one can enrich each theory by the results from the other theory. In the present study, we have only focused on the generalization of rough sets using the results from modal logics. As a further research topic, one may use the results from rough sets to interpret and extend modal logics. For example, Orlowska proposed a logic of indiscernibility by using the results from rough set theory.[22] Another research topic is the generalization of present study by using fuzzy similarity relation instead of ordinary binary relation. Some important results have been reported by Nakamura.[20, 21] It will be worthwhile to carry out more study along this line.

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