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Generalized Rough Set Models

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1 Introduction

Since Pawlak [43] proposed the rough set theory in early eighties, many proposals have been made for generalizing and interpreting rough sets [3, 4, 19, 27, 37, 38, 39, 46, 47, 51, 52, 54, 55, 63, 65, 66, 67, 68, 69, 70, 71, 72, 83, 92, 103, 104, 105]. Extensive studies have been carried out to compare the theory of rough sets and other theories of uncertainty, such as fuzzy sets [5, 12, 21, 56, 82, 85, 91], modal logic [31, 35, 38, 100], conditional events [33, 34], and Dempster-Shafer theory of evidence [16, 62, 64, 93]. A recent review of standard and generalized rough set models, and their relationships to other theories, is given by Yao *et al* [99].

Haack [17] classified non-classical logics into roughly two groups. A non-classical logic is a deviation of classical two-valued logic, i.e., a deviant logic, if the two logics have the same logical vocabulary but different axioms or rules. Many-valued logics may be viewed as deviant logics. A non-classical logic is an extension, i.e., an extended logic, if it adds new vocabulary along with new axioms or rules for the new vocabulary. Modal logics may be viewed as extended logics. Classical set-theoretic operators reflect the corresponding logic connectives in classical two-valued logic [21]. For non-classical set theories, Klir [21] compared the roles played by non-classical logics, such as many-valued logics and modal logics, for interpreting fuzzy sets and rough sets. Using the similar argument, Yao [90] suggested that non-classical set theories may be viewed as deviations and extensions of classical set theory.

For the interpretation of rough set theory, Yao [86] pointed out that at least two views may be used, the operator-oriented view and set-oriented view. The operator-oriented view interprets rough set theory as an *extension* of set theory with two additional unary set-theoretic operators. In other words, in rough set theory we study the system $(2^U, \sim, \underline{apr}, \overline{apr}, \cap, \cup)$, where 2^U is the power set of a finite and nonempty set U, and $(2^U, \sim, \cap, \cup)$ is the standard set algebra. In this case, the meaning of standard set-theoretic operators is unchanged. On the other hand, the set-oriented view focuses on a system $(R(U), \neg, \sqcap, \sqcup)$, where $R(U)$ is the set of all rough sets defined on U. Under this view, one must define rough set operators $\neg$, $\sqcap$, and $\sqcup$ by modifying standard set-theoretic operator,

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which produces a *deviation* of set theory. No additional set-theoretic operator is introduced.

For the formulation of rough set theory, roughly two approaches may be taken, the constructive and algebraic approaches [89]. Consider a constructive approach in which one starts from a binary relation and defines a pair of lower and upper approximation operators. The approximation operators have a similar interpretation as that of necessity and possibility operators in modal logic. The relationships between rough sets and modal logics have been studied extensively by many authors [1, 2, 10, 24, 25, 30, 31, 32, 35, 36, 37, 38, 40, 44, 45, 57, 58, 59, 74, 75, 76, 79]. Yao and Lin [92] introduced and classified generalized rough set models. Various classes of rough set models are proposed and examined using different types of binary relations. Consider now an algebraic (axiomatic) approach in which one defines a pair of dual approximation operators and states axioms that must be satisfied by the operators. Various classes of rough set algebras are characterized by different sets of axioms. Such an axiomatic characterization of approximation operators has been investigated by a number of authors [28, 77, 83, 86, 89].

In this paper, we focus mainly on rough set algebras characterized by a pair of lower and upper approximation operators, and rough sets defined by rough membership functions. We will not discuss other related studies on rough sets, except for giving a few references. Iwinski [19] and Pawlak [43] defined a rough set to be a family of subsets of a universe that have the same lower and upper approximations. With such a notion of rough sets, many researchers established some important connections between rough set theory and other algebraic systems. Pomykala and Pomykala [53] used Stone algebras as the algebraic system for modeling rough sets, in which a psedocomplement operator was introduced. By introducing a dual psedocomplement, Comer [7, 8] suggested that in fact the regular double Stone algebras may be used. For an exposition of the algebraic connections between rough sets, rough relation algebras, Nelson algebras, three-valued Lukasiewicz algebras, and related structures, one may consult papers by Cattaneo [3, 4], Düntsch [13], and Pagliani [40, 41].

This paper is a sequel of the review by Yao *et al* [99]. We present some new results on generalized rough set models from both constructive and algebraic points of views. The rest of the paper is organized as follows. From Section 2 to 5, we concentrate on an operator-oriented view of rough sets. In Section 2, we review a constructive mothed of rough set theory, which builds approximation operators from binary relations. In Section 3, we introduce and examine alternative representations of approximation operators, and transformations from one to another [86]. In Section 4, we present an algebraic method of rough set theory. Axioms on approximation operators are studied. In Section 5, we study the connections between the theory of rough sets and other related theories of uncertainty [16, 61, 62, 64, 80, 93]. Two special classes of rough set models are studied. They are related to belief and plausibility functions, and necessity and possibility functions, respectively. Section 6 deals with a set-oriented view of rough sets based on probabilistic rough set models [49, 97, 104] and rough membership functions [48]. It enables us to draw connections between rough sets and

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fuzzy sets. The notion of interval rough membership functions is introduced.

For simplicity, we restrict our discussion to finite and nonempty universes. Some of the results may not necessarily hold for infinite universes.

2 Construction of Rough Set Models

Let U denote a finite and nonempty set called the universe, and let $E \subseteq U \times U$ denote an equivalence relation on U, namely, E is reflexive, symmetric, and transitive. The pair $apr = (U, E)$ is called a Pawlak approximation space. The equivalence relation E partitions the set U into disjoint subsets. Such a partition of the universe is a quotient set of U, written U/E. Elements of U/E are called the elementary sets. The empty set $\emptyset$ and the union of one or more elementary sets are called definable, observable, measurable, or composed sets. The family of all definable sets is denoted by $\sigma(U/E)$. It is an σ-algebra of subsets of U generated by the family of equivalence classes U/E. In addition, U/E is the basis of the σ-algebra $\sigma(U/E)$.

The equivalence relation and the induced equivalence classes may be regarded as the available information or knowledge about the objects under consideration. For two elements $x, y \in U$, if xEy, we say that x and y are indistinguishable. Equivalence classes are the basic building blocks for the representation and approximation of any subset of the universe. They generate the σ-algebra $\sigma(U/E)$. All subsets of U must be represented using the elements of $\sigma(U/E)$. More specifically, for any subset $A \subseteq U$, the greatest definable set contained in A is called the lower approximation of A, written $\underline{apr}A$, while the least definable set containing A is called the upper approximation of A, written $\overline{apr}A$. They can be expressed as:

$$\begin{aligned} \underline{apr}A &= \bigcup \{X \mid X \in \sigma(U/E), X \subseteq A\}, \\ \overline{apr}A &= \bigcap \{X \mid X \in \sigma(U/E), X \supseteq A\}. \end{aligned} \tag{1}$$

The set A lies between its lower and upper approximations. By definition, a definable set has the same lower and upper approximations. In terms of equivalence classes, lower and upper approximations can be expressed by:

$$\begin{aligned} \underline{apr}A &= \bigcup_{[x]_E \subseteq A} [x]_E, \\ \overline{apr}A &= \bigcup_{[x]_E \cap A \neq \emptyset} [x]_E, \end{aligned} \tag{2}$$

where

$$[x]_E = \{y \mid xEy\}, \tag{3}$$

is the equivalence class containing x. The lower approximation $\underline{apr}A$ is the union of equivalence classes which are subsets of A. The upper approximation $\overline{apr}A$

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is the union of equivalence classes which have a nonempty intersection with A. They can be equivalently defined by:

$$\underline{apr}A = \{x \mid [x]_E \subseteq A\},$$
$$\overline{apr}A = \{x \mid [x]_E \cap A \neq \emptyset\}. \tag{4}$$

One may interpret $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^U$ as two unary set-theoretic operators. They are called approximation operators, and the system $(2^U, \sim, \underline{apr}, \overline{apr}, \cap, \cup)$ is called a Pawlak rough set algebra [86]. It is an extension of the set algebra $(2^U, \sim, \cap, \cup)$.

The notion of approximation operators can be generalized by considering an arbitrary binary relation. Let $R \subseteq U \times U$ be a binary relation on the universe. The pair $apr = (U, R)$ is called an approximation space. Given two elements $x, y \in U$, if xRy, we say that y is R-related to x, x is a predecessor of y, and y is a successor of x. From a binary relation R, for an element $x \in U$ we define its successor neighborhood as [88, 92]:

$$R_s(x) = \{y \in U \mid xRy\}. \tag{5}$$

The notion of successor neighborhoods can be easily extended to any subset $A \subseteq U$ as follows:

$$R_s(A) = \bigcup_{x \in A} R_s(x). \tag{6}$$

For the empty set, we define $R_s(\emptyset) = \emptyset$. For any subset A of the universe, we define a pair of lower and upper approximations by replacing the equivalence class $[x]_R$ with the successor neighborhood $R_s(x)$ in equation (4):

$$\underline{apr}A = \{x \mid R_s(x) \subseteq A\},$$
$$\overline{apr}A = \{x \mid R_s(x) \cap A \neq \emptyset\}. \tag{7}$$

An element x in U belongs to the lower approximation $\underline{apr}A$ if x's neighborhood is contained in A, and x belongs to $\overline{apr}A$ is x's neighborhood has a nonempty intersection with A. That is, $\underline{apr}A$ consists of those elements whose R-related elements are *all* in A, and $\overline{apr}A$ consists of those elements such that at least *one* of whose R-related elements is in A. They can therefore be equivalently defined by:

$$\underline{apr}A = \{x \in U \mid \text{for all } y \in U,\ xRy \text{ implies } y \in A\},$$
$$\overline{apr}A = \{x \in U \mid \text{there exists } y \in U \text{ such that } xRy \text{ and } y \in A\}. \tag{8}$$

This interpretation is closely related to the definition of the necessity and possibility operators in the study of modal logic [6, 92].

Although no restriction is imposed on the binary relation, the approximation operators satisfy the following properties: for subsets $A, B \subseteq U$,

(L0) $\underline{apr}A = \sim\overline{apr} \sim A$,

(L1) $\underline{apr}U = U$,

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$$
\begin{array}{ll}
\text{(L2)} & \underline{apr}(A \cap B) = \underline{apr}A \cap \underline{apr}B, \\
\text{(L3)} & \underline{apr}(A \cup B) \supseteq \underline{apr}A \cup \underline{apr}B, \\
\text{(L4)} & A \subseteq B \Longrightarrow \underline{apr}A \subseteq \underline{apr}B, \\
\text{(U0)} & \overline{apr}A = \sim \underline{apr} \sim A, \\
\text{(U1)} & \overline{apr}\emptyset = \emptyset, \\
\text{(U2)} & \overline{apr}(A \cup B) = \overline{apr}A \cup \overline{apr}B, \\
\text{(U3)} & \overline{apr}(A \cap B) \subseteq \overline{apr}A \cap \overline{apr}B, \\
\text{(U4)} & A \subseteq B \Longrightarrow \overline{apr}A \subseteq \overline{apr}B.
\end{array}
$$

Properties (L0) and (U0) state that two approximations are dual to each other. Properties with the same number may be regarded as dual properties. They are not independent properties. Properties (L1) and (L2) form a set of independent properties of lower approximation operator $\underline{apr}$, and (Ul) and (U2) form a set of independent properties of upper approximation operator $\overline{apr}$.

Within the proposed framework, one may study rough set models constructed from special types of binary relations [92]. A binary relation R is a serial relation if for all $x \in U$ there exists $y \in U$ such that xRy. A relation is a reflexive relation if for all $x \in U$ the relationship xRx holds. A relation is symmetric relation if for all $x, y \in U$, xRy implies yRx holds. A relation is a transitive relation if for all $x, y, z \in U$, xRy and yRz imply xRz. A relation is Euclidean if for all $x, y, z \in U$, xRy and xRz imply yRz. With respect to these types of binary relations, approximation operators have additional properties:

$$
\begin{array}{rll}
\text{serial} & \text{(D)} & \underline{apr}A \subseteq \overline{apr}A, \\
\text{reflexive} & \text{(T)} & \underline{apr}A \subseteq A, \\
 & \text{(T}'\text{)} & A \subseteq \overline{apr}A, \\
\text{symmetric} & \text{(B)} & A \subseteq \underline{apr}\ \overline{apr}A, \\
 & \text{(B}'\text{)} & \overline{apr}\ \underline{apr}A \subseteq A, \\
\text{transitive} & \text{(4)} & \underline{apr}A \subseteq \underline{apr}\ \underline{apr}A, \\
 & \text{(4}'\text{)} & \overline{apr}\ \overline{apr}A \subseteq \overline{apr}A, \\
\text{Euclidean} & \text{(5)} & \overline{apr}A \subseteq \underline{apr}\ \overline{apr}A, \\
 & \text{(5}'\text{)} & \overline{apr}\ \underline{apr}A \subseteq \underline{apr}X.
\end{array}
$$

These results follow from the definition of approximation operators. They are parallel to that of necessity and possibility in modal logic. For easy comparisons, we adopt the same labeling system used by Chellas [6] for modal logic.

A *serial* rough set model is obtained from a serial binary relation. Given the first three properties of approximation operators, from property (D) we have: for any $A \subseteq U$,

$$
\begin{aligned}
\underline{apr}A \subseteq \overline{apr}A &\Longleftrightarrow \underline{apr}\ A \cap \sim \overline{apr}A = \emptyset \\
&\Longleftrightarrow \underline{apr}A \cap \underline{apr} \sim A = \emptyset \\
&\Longleftrightarrow \underline{apr}(A \cap \sim A) = \emptyset \\
&\Longleftrightarrow \underline{apr}\emptyset = \emptyset. \qquad (9)
\end{aligned}
$$

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Similarly, we have:

$$\underline{apr}A \subseteq \overline{apr}A \Longleftrightarrow \overline{apr}U = U. \tag{10}$$

Therefore, (D) can be replaced by two more familiar axioms:

$$\text{(L5)} \qquad \underline{apr}\emptyset = \emptyset,$$
$$\text{(U5)} \qquad \overline{apr}U = U.$$

The pair of approximation operators of a serial rough set model is referred to as an interval structure [81]. Serial rough set models can be used to establish a connection between rough set theory and belief functions [81, 93]. Many other classes of rough set models can be constructed based on various binary relations [92]. The class obtained from transitive and connected binary relations is of interest. This class can be used to establish a connection between rough set theory and the class of consonant belief and plausibility functions, i.e., necessity and possibility functions [22, 97]. A detailed discussion on this topic will be presented in Section 5.

By following the same procedure, one may construct more approximation operators from the predecessor neighborhoods, or combinations of predecessor and successor neighborhoods defined by binary relations [88]. One may also introduce new approximation operators by the composition of existing approximation operators [89].

Cattaneo [4] discussed another method for the construction of approximation operators from binary relations. He used an irreflexive and symmetric binary relation called discernibility or preclusivity relation. Such a relation is in fact the complement of a reflexive and symmetric relation called a compatibility or tolerance relation [68, 92, 103]. Its relationship to our formulation can therefore be established [4]. Let $\#$ denote an irreflexive and symmetric relation on U. From $\#$, the preclusive orthocomplement of a subset $A \subseteq U$ is defined by:

$$A^{\#} = \{x \in U \mid \text{ for all } y \in A, x\#y\}. \tag{11}$$

By combining $^{\#}$ and the standard set complement operator ^{c}, one can define two pairs of approximation operators, $(H^{c\#}, H^{\#c})$ and $(H^{bb}, H^{\#\#})$, where $H^{b} = H^{c\#c}$. They are related to each other by the chain of inclusion:

$$H^{c\#} \subseteq H^{bb} \subseteq H^{\#\#} \subseteq H^{\#c}. \tag{12}$$

The complement of the relation $\#$ is given by $R^{\#} = U \times U - \#$, which is a reflexive and symmetric relation. Let $\underline{apr}$ and $\overline{apr}$ be the approximation operators defined by using successor neighborhoods of $R^{\#}$. We immediately have the following relationships:

$$A^{c\#} = \underline{apr}A, \quad A^{\#c} = \overline{apr}A,$$
$$A^{bb} = \overline{apr}\,\underline{apr}A, \quad A^{\#\#} = \underline{apr}\,\overline{apr}A. \tag{13}$$

The second pair of approximation operators can be obtained from compositions of the first pair.

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3 Alternative Representations of Approximation Operators

In the last section, approximation operators are constructed and interpreted based on binary relations. Two additional representations of approximation operators will be presented in this section by extending the results of Yao [87]. These notions have been studied by Pawlak [47], Skowron [63], and Skowron and Grzymala-Buss [64] in the contexts of Pawlak rough set models and Pawlak information systems [42].

3.1 Upper Approximation Distributions

Based on property (U2) and the finiteness property of the universe, the upper approximation of a set can be computed from the upper approximations of its singleton subsets, namely, for any $A \subseteq U$, we have:

$$\overline{apr} A = \bigcup_{x \in A} \overline{apr}\{x\}. \tag{14}$$

If $A = \emptyset$, we define $\overline{apr}\emptyset = \emptyset$. This suggests another way to represent an upper approximation operator. We define a function $h : U \longrightarrow 2^U$,

$$h(x) = \overline{apr}\{x\}. \tag{15}$$

The upper approximation of any subset $A \subseteq U$ can be expressed as:

$$\overline{apr} A = \bigcup_{x \in A} h(x). \tag{16}$$

The function h plays a similar role as that of probability distribution in probability theory, and possibility distribution in possibility theory [100]. We call the function h an upper approximation distribution. Property (U2) shows that an upper approximation operator is additive. An upper approximation operator is therefore an additive extension of an upper approximation distribution, and an upper approximation distribution is a projection of an upper approximation on singleton subsets of the universe.

The notion of upper approximation distribution can be interpreted using binary relations. For a binary relation R on U, its converse R^{-1} is defined by:

$$yR^{-1}x \Longleftrightarrow xRy. \tag{17}$$

For an element $y \in U$, the set of R^{-1}-related elements (i.e., conversely R-related elements) is given by:

$$R_s^{-1}(y) = \{x \mid yR^{-1}x\} = \{x \mid xRy\} = \{x \mid y \in R_s(x)\}. \tag{18}$$

On the other hand, the upper approximation of a singleton subset $\{y\}$ is:

$$\overline{apr}\{y\} = \{x \mid R_s(x) \cap \{y\} \neq \emptyset\} = \{x \mid y \in R_s(x)\}. \tag{19}$$

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By combining these results, we have:

$$h(y) = \{x \mid y \in R_s(x)\} = R_s^{-1}(y). \tag{20}$$

That is, $h(y)$ is exactly the set of R^{-1}-related elements of y. This implies that $x \in h(y)$ if and only if $y \in R_s(x)$. From the upper approximation distribution, a binary relation can be defined by:

$$R_s(x) = \{y \mid x \in h(y)\}. \tag{21}$$

Such a relationship can be extended to binary relations and approximation operators.

In defining rough set algebras, we do not impose any constraint on the binary relation. Consequently, there is not any constraint on an upper approximation distribution. Given an arbitrary function $h : U \longrightarrow 2^U$, we can define an upper approximation operator by using equation (16). By definition, the upper approximation operator satisfies properties (U0)-(U4).

3.2 Basic Set Assignments

A binary relation R associates each element of U with its successor neighborhood $R_s(x) \subseteq U$. By collecting the elements associated with the same subset of U, we define a function $m : 2^U \longrightarrow 2^U$:

$$m(A) = \{x \mid R_s(x) = A\}. \tag{22}$$

An element $x \in m(A)$ if and only if the set of R-related elements of x is exactly A. Each element of the universe must be associated with one and only one subset of the universe. This implies that m must satisfy the axioms: for all $A, B \subseteq U$,

$$\begin{array}{ll} \text{(A1)} & \bigcup_{A \subseteq U} m(A) = U, \\ \text{(A2)} & A \neq B \Longrightarrow m(A) \cap m(B) = \emptyset. \end{array}$$

The set $m(A)$ can be considered as an equivalence class of the following equivalence relation on U:

$$x \equiv y \Longleftrightarrow R_s(x) = R_s(y). \tag{23}$$

Under $\equiv$, two elements of U are considered to be equivalent if and only if their R-related elements are the same, or equivalently they have the same successor neighborhood. Conversely, from m the relation R can be recovered by:

$$R_s(x) = A, \qquad \text{for } x \in m(A). \tag{24}$$

By properties (A1) and (A2), for every element x in U there exists a unique set $A \subseteq U$ such that $x \in m(A)$. Therefore, this definition is properly defined.

For an arbitrary subset $A \subseteq U$, consider an element $x \in \underline{apr}A$. By the definition of $\underline{apr}$, we have $R_s(x) \subseteq A$. This implies that there must exist a unique subset $B \subseteq A$ such that $B = R_s(x)$, namely, $x \in m(B)$. On the other

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hand, for an arbitrary subset $B \subseteq A$, consider an element $x \in m(B)$. By the definition of m, we have $R_s(x) = B \subseteq A$. This implies that $x \in \underline{apr}A$. From the above argument, it follows that $\underline{apr}$ can be expressed in terms of m:

$$\underline{apr}A = \bigcup_{B \subseteq A} m(B). \tag{25}$$

The upper approximation can be expressed as:

$$\overline{apr}A = \bigcup_{B \cap A \neq \emptyset} m(B), \tag{26}$$

which can be derived either from definitions of m and $\overline{apr}$, or from the relationship $\overline{apr}A = \sim \underline{apr} \sim A$.

Function m can be computed from a lower approximation operator. According to equation (25) and the definition of m, if $x \in m(A)$, one can conclude that $x \in \underline{apr}A$ and $x \notin \underline{apr}B$ for any proper subset $B \subset A$. That is, $m(A) \subseteq \underline{apr}A - \bigcup_{B \subset A} \underline{apr}B$. For any element $x \in \underline{apr}A$, there is a unique subset $B \subseteq A$ such that $x \in m(B)$, which implies $x \in \underline{apr}B$. Therefore, if $x \in \underline{apr}A - \bigcup_{B \subset A} \underline{apr}B$, it must be the case that $x \in m(A)$. That is, $\underline{apr}A - \bigcup_{B \subset A} \underline{apr}B \subseteq m(A)$. Hence, $m(A)$ can be expressed through the lower approximation operator as:

$$m(A) = \underline{apr}A - \bigcup_{B \subset A} \underline{apr}B. \tag{27}$$

Function m plays a similar role as that of basic probability assignment in the theory of belief functions. In terms of a binary relation R, an element $x \in m(A)$ is exactly related to the set A. This is similar to the interpretation of basic probability assignment in the theory of belief functions [29]. For this reason, m is called a basic set assignment [81, 100].

A subset with $m(A) \neq \emptyset$ is called a focal set. The family of focal sets is given by:

$$F = \{A \mid A \subseteq U, m(A) \neq \emptyset\}. \tag{28}$$

From images of F by m, we define a family of subsets of U:

$$M = \{m(A) \mid A \in F\}. \tag{29}$$

By properties (A1) and (A2), one can see that M is a partition of the universe, which is in fact the quotient set defined by the equivalence relation $\equiv$, namely $M = U/\equiv$. Elements of M are called elementary sets. The empty set $\emptyset$ and the unions of one or more elementary sets are called observable sets [93]. The family of all observable sets formed from M is denoted by $\sigma(M)$, which is the σ-algebra generated by M. The family of subsets M is the basis of $\sigma(M)$. With respect to a binary relation R, for a focal set $A \in F$, we have $R_s(m(A)) = R_s(x) = A$ for

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any $x \in m(A)$. Using these notions, the lower and upper approximations have similar interpretations as in the Pawlak rough set algebra, namely,

$$\underline{apr}A = \bigcup\{X \mid X \in \sigma(M), R_s(X) \subseteq A\},$$
$$\overline{apr}A = \bigcap\{X \mid X \in \sigma(M), \sim R_s(\sim X) \supseteq A\}. \tag{30}$$

The lower approximation of A is the greatest observable set in $\sigma(M)$ whose successor neighborhood is contained in A, and the upper approximation is the least observable set in $\sigma(M)$ such that the complement of its complement's successor neighborhood contains A. Similarly, in terms of equivalence classes of $\equiv$, lower and upper approximations can be expressed by:

$$\underline{apr}A = \bigcup_{R_s([x]_\equiv)\subseteq A} [x]_\equiv,$$
$$\overline{apr}A = \bigcup_{R_s([x]_\equiv)\cap A\neq\emptyset} [x]_\equiv, \tag{31}$$

where $[x]_\equiv = \{y \mid x \equiv y\} = \{y \mid R_s(x) = R_s(y)\}$ is the equivalence class containing x. Results given by equations (30) and (31) are in parallel to that of equations (1) and (2).

3.3 Related Works

In order to obtain more insights into the alternative representations, we first examine them in the context of a Pawlak rough set model, and then summarize some of earlier studies [47, 64, 81, 87].

Consider a Pawlak rough set model constructed from an equivalence relation E. From the properties of an equivalence relation, we immediately have the following interesting results:

$$\begin{array}{ll}
\text{(i)} & \forall x, y \in U[xEy \Longleftrightarrow x \equiv y],\\
\text{(ii)} & F = M = U/E = U/\equiv, \quad \sigma(M) = \sigma(U/E),\\
\text{(iii)} & \forall x \in U[E_s(x) = [x]_E],\\
\text{(iv)} & \forall A \in \sigma(U/E)[E_s(A) = A],\\
\text{(v)} & h(x) = [x]_E,\\
\text{(vi)} & m(A) = \begin{cases} A & A \in U/E,\\ \emptyset & otherwise.\end{cases}
\end{array}$$

They show that different notions coincide in the Pawlak rough set model. By applying (iii) into equation (7), we have equation (4). Applying (ii) and (iv) into equation (30) results in equation (1). By combining (v) and equation (16), we have another definition of the upper approximation:

$$\overline{apr}A = \bigcup_{x\in A} [x]_E. \tag{32}$$

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

This definition was used by some authors [8, 14]. Finally, by applying (vi) into equations (25) and (26), or by combining (ii), (v), and equation (31), we have equation (2). In summary, our formulation of rough sets provides a more general framework. Notions, such as successor neighborhoods, upper approximation distributions, and basic set assignments, are useful in understanding generalized rough set models, although they become trivial concepts in the Pawlak rough set model.

Pawlak [47] interpreted an upper approximation distribution as being an information function and used the symbol $I(x)$, i.e., $I(x) = h(x)$. The information about x is given by $I(x)$. Every element $y \in I(x)$ is considered to be *indiscernible* from x with respect to the information I. In order to use such an interpretation, one has to impose certain constraints on the information function I, such as $x \in I(x)$.

Skowron and Grzymala-Buss [64] implicitly used the notion of basic set assignment in study of connections between rough sets and belief functions. In a special case, they used the mapping as given by (vi). In the general case, they adopted notions similar to interval structures, in which two universes and a relation between them are used [81]. One universe is a set of objects and the other universe is a set of decision classes. The relationship between elements of the two universes are given by a Pawlak information system. For details on such a formulation, one may read papers by Skowron and Grzymala-Buss [64], Wong *et al.* [81], and Yao and Lingras [93].

3.4 An Example

A simple example is presented to illustrate the basic ideas developed in the previous subsections.

Consider a universe $U = \{a, b, c\}$. Suppose a binary relation R on U is given by:

$$aRa, \quad bRb, \quad aRb, \quad bRa, \quad cRb.$$

The successor neighborhoods are defined by:

$$R_s(a) = \{a, b\}, \quad R_s(b) = \{a, b\}, \quad R_s(c) = \{b\}.$$

The converse relation of R is given by:

$$aR^{-1}a, \quad bR^{-1}b, \quad bR^{-1}a, \quad aR^{-1}b, \quad bR^{-1}c.$$

The corresponding successor neighborhoods are:

$$R_s^{-1}(a) = \{a, b\}, \quad R_s^{-1}(b) = U, \quad R_s^{-1}(c) = \emptyset.$$

Based on R, one can define an equivalence relation $\equiv$ by:

$$a \equiv a, \quad b \equiv b, \quad a \equiv b, \quad b \equiv a, \quad c \equiv c,$$

which induces a partition of the universe U, namely, $M = \{\{a, b\}, \{c\}\}$. The σ-algebra generated by M is given by:

$$\sigma(M) = \{\emptyset, \{a, b\}, \{c\}, U\}.$$

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

Every subset of the universe is approximated by two elements of $\sigma(M)$.

The binary relation produces the pair of lower and upper approximation operators:

$$\begin{array}{ll}
\underline{apr}\emptyset = \emptyset, & \overline{apr}\emptyset = \emptyset, \\
\underline{apr}\{a\} = \emptyset, & \overline{apr}\{a\} = \{a, b\}, \\
\underline{apr}\{b\} = \{c\}, & \overline{apr}\{b\} = U, \\
\underline{apr}\{c\} = \emptyset, & \overline{apr}\{c\} = \emptyset, \\
\underline{apr}\{a, b\} = U, & \overline{apr}\{a, b\} = U, \\
\underline{apr}\{a, c\} = \emptyset, & \overline{apr}\{a, c\} = \{a, b\}, \\
\underline{apr}\{b, c\} = \{c\}, & \overline{apr}\{b, c\} = U, \\
\underline{apr}U = U, & \overline{apr}U = U.
\end{array}$$

The upper approximation distribution is defined:

$$h(a) = \{a, b\}, \quad h(b) = U, \quad h(c) = \emptyset.$$

The basic set assignment is:

$$\begin{array}{llll}
m(\emptyset) = \emptyset, & m(\{a\}) = \emptyset, & m(\{b\}) = \{c\}, & m(\{c\}) = \emptyset, \\
m(\{a, b\}) = \{a, b\}, & m(\{a, c\}) = \emptyset, & m(\{b, c\}) = \emptyset, & m(U) = \emptyset.
\end{array}$$

The set of focal sets is:

$$F = \{\{b\}, \{a, b\}\},$$

which defines the following family of subsets of U:

$$M = \{\{c\}, \{a, b\}\},$$

which is in fact the partition induced by the equivalence relation $\equiv$.

From this example, one can easily verify the relationships between a binary relation, successor neighborhoods, a pair of approximation operators, an upper approximation distribution, and a basic set assignment. Transformations from one to any other are summarized in Table 1, where a pair of lower and upper approximations is represented by **L** and **H**.

4 Axiomatic Characterization of Rough Set Models

In the construction of approximation operators, we started from an arbitrary binary relation by treating it as a primitive notion. Such a constructive approach is commonly used in the study of rough set theory [89]. Alternatively, one may define a pair of approximation operators axiomatically by using certain axioms, without explicitly referring to a binary relation. In this case, the approximation operators are used as primitive notions. We study an algebra $(2^U, \sim, \mathbf{L}, \mathbf{H}, \cap, \cup)$, where **L** and **H** are unary set-theoretic operators referred to as approximation operators. An important advantage of the axiomatic or algebraic method is that one may gain more insights into the structures of lower and upper approximation operators. An axiomatic characterization of approximation operators is presented in this section.

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

Definition 1. A mapping $\mathbf{L} : 2^U \longrightarrow 2^U$ is called a lower approximation operator if it obeys two axioms: for $A, B \subseteq U$,

$$\text{(L1)} \qquad \mathbf{L}U = U,$$
$$\text{(L2)} \qquad \mathbf{L}(A \cap B) = \mathbf{L}A \cap \mathbf{L}B.$$

A mapping $\mathbf{H} : 2^U \longrightarrow 2^U$ is called an upper approximation operator if it obeys two axioms:

$$\text{(U1)} \qquad \mathbf{H}\emptyset = \emptyset,$$
$$\text{(U2)} \qquad \mathbf{H}(A \cup B) = \mathbf{H}A \cup \mathbf{H}B.$$

Axioms (L1) and (U1) give the conditions on the approximations of two special subsets of U, the entire set U and the empty set $\emptyset$. The lower approximation of the universe is itself and the upper approximation of the empty set is the empty set. Axioms (L2) and (U2) may be understood as the distributivity of a lower approximation operator over set intersection, and of an upper approximation operators over set union. They show that a lower approximation is multiplicative and an upper approximation is additive [83, 84].

Definition 2. Operators $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ are said to be dual if they are related to each other by:

$$\text{(L0)} \qquad \mathbf{L}A = \sim\mathbf{H} \sim A,$$
$$\text{(U0)} \qquad \mathbf{H}A = \sim\mathbf{L} \sim A.$$

By the duality of lower and upper approximation operators, we only need to define one operator. For example, if we define an upper approximation operator $\mathbf{H}$, its dual lower approximation operator $\mathbf{L}$ may be considered as an abbreviation of $\sim\mathbf{H}\sim$. In parallel to the constructive approach, we introduce notions of upper approximation distributions and basic set assignments. They are alternative representations of approximation operators.

Definition 3. A mapping $h : U \longrightarrow 2^U$ without any constraints is called an upper approximation distribution. A mapping $m : 2^U \longrightarrow 2^U$ is called a basic set assignment if it obeys two axioms: for $A, B \subseteq U$,

$$\text{(A1)} \qquad \bigcup_{A \subseteq U} m(A) = U,$$
$$\text{(A2)} \qquad A \neq B \Longrightarrow m(A) \cap m(B) = \emptyset.$$

Table 1 summarizes transformations between approximation operators, a binary relation, an upper approximation distribution, and a basic set assignment. From these transformations, we have the following useful results.

Theorem 4. *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *are dual approximation operators. The following conditions are equivalent:*

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

(a). **L** *is a lower approximation operator satisfying axioms* (L1) *and* (L2)*;*
(b). **H** *is an upper approximation operator satisfying axioms* (U1) *and* (U2)*;*
(c). *There exists an upper approximation distribution* $h : U \longrightarrow 2^U$ *such that for all subsets* $A \subseteq U$, $\mathbf{H}(A) = \bigcup_{x \in A} h(x)$*;*
(d). *There exist a binary relation* R *and the corresponding approximation space* $apr = (U, R)$ *such that for all* $A \subseteq U$, $\mathbf{H}(A) = \overline{apr}(A)$*;*
(e). *There exists a basic set assignment* $m : 2^U \longrightarrow 2^U$ *satisfying axioms* (A1) *and* (A2) *such that for all* $A \subseteq U$, $\mathbf{L}(A) = \bigcup_{B \subseteq A} m(B)$.

Equivalence of (a) to (c) can be easily established. Equivalence of (b) and (d) was proved by Yao [86, Theorem 3, page 298], and equivalence of (a) and (e) follows from the discussion on basic set assignments in Section 3.2.

According to Theorem 4, axioms (L1), (L2), (U1), and (U2) are considered to be the basic axioms of approximation operators, axioms (A1) and (A2) are the basic axioms of basic set assignment. This leads to the following definition of rough set algebra.

Definition 5. An algebra $(2^U, \sim, \mathbf{L}, \mathbf{H}, \cap, \cup)$ is called a rough set algebra if **L** and **H** are dual approximation operators satisfying axioms (L1), (L2), (U1), and (U2).

The definition of basic set assignment is weaker than the proposal of Wong *et al* [81]. They used an additional axiom:

$$\text{(A3)} \qquad m(\emptyset) = \emptyset.$$

In the construction of rough set models, no constraint is imposed on binary relations. There may be elements that are not related to any element in U, i.e., $R_s(x) = \emptyset$. Such elements belong to $m(\emptyset)$. Therefore, axiom (A3) is dropped. From the view point of approximation operators, this is equivalent to drop axioms (L5) and (U5) used by Wong *et al* [81] in the study of interval structures. The notion of rough set algebras is more general than interval structures. In order to include axiom (A3), one must use a serial relation. Following the same arguments, we may study different classes of binary relations and axioms on various representations of approximation operators. Their relationships are summarized in Tables 2 and 3. From these tables, one may obtain a number of theorems in parallel to Theorem 4. For instance, with respect to a serial binary relation, we have the following theorem.

Theorem 6. *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *are dual approximation operators. The following conditions are equivalent:*

(a). **L** *is a lower approximation operator satisfying axioms* (L1), (L2)*, and* (L5)*;*
(b). **H** *is an upper approximation operator satisfying axioms* (U1), (U2)*, and* (U5)*;*
(c). *There exists an upper approximation distribution* $h : U \longrightarrow 2^U$ *satisfying axiom* (h1) *such that for all subsets* $A \subseteq U$, $\mathbf{H}(A) = \bigcup_{x \in A} h(x)$*;*

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

(d). *There exist a serial binary relation R and the corresponding approximation space $apr = (U, R)$ such that for all $A \subseteq U$, $\mathbf{H}(A) = \overline{apr}(A)$;*
(e). *There exists a basic set assignment $m : 2^U \longrightarrow 2^U$ satisfying axioms* (A1), (A2), *and* (A3) *such that for all $A \subseteq U$, $\mathbf{L}(A) = \bigcup_{B \subseteq A} m(B)$.*

A Pawlak rough set model is constructed by an equivalence relation. In this case, we have the corresponding theorem.

Theorem 7. *Suppose $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ are a pair of dual approximation operators. The following conditions are equivalent:*

(a). $\mathbf{L}$ *is a lower approximation operator satisfying axioms* (L1), (L2), (T), (B), *and* (4)*;*
(b). $\mathbf{H}$ *is an upper approximation operator satisfying axioms* (U1), (U2), (T′), (B′), *and* (4′)*;*
(c). *There exists an upper approximation distribution $h : U \longrightarrow 2^U$ satisfying axiom* (h2), (h3), *and* (h4) *such that for all subsets $A \subseteq U$, $\mathbf{H}(A) = \bigcup_{x \in A} h(x)$;*
(d). *There exist an equivalence relation E and the corresponding approximation space $apr = (U, E)$ such that for all $A \subseteq U$, $\mathbf{H}(A) = \overline{apr}(A)$;*
(e). *There exists a basic set assignment $m : 2^U \longrightarrow 2^U$ satisfying axioms* (A1), (A2), (A4), (A5), *and* (A6) *such that for all $A \subseteq U$, $\mathbf{L}(A) = \bigcup_{B \subseteq A} m(B)$.*

The equivalence of (a), (b), and (d) has been discussed by many authors [14, 28, 78, 86]. The equivalence of other conditions can be easily proved. Statement (e) in the previous theorems in fact establishes a connection between constructive and algebraic methods of the theory of rough sets. That is, certain axioms of approximation operators guarantee the existence of a special class of binary relations producing the same operators, and vice versa.

In comparison with studies on constructive methods, there is not enough attention on algebraic methods [89]. Some earlier algebraic studies are briefly summarized below. Zakowski [102] studied a set of axioms on approximation operators, including axioms such as (L2), (U2), (T), and (T′). A problem with Zakowski's axiomatization is that an equivalence relation is explicitly used. One axiom states that the lower and upper approximations of definable sets are the sets themselves. The notion of definable sets must be defined using an equivalence relation. Comer [7, 8] investigated axioms on approximation operators in relation to cylindric algebras. The investigation is made within the context of Pawlak information systems [42]. Lin and Liu [28] studied axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. The similar result was stated earlier by Wiweger [78]. All those studies are restricted to Pawlak rough set algebra defined by equivalence relations. Wybraniec-Skardowska [83, 84] examined many axioms on various classes of approximation operators. Yao [86, 89] extended axiomatic approach to rough set algebras constructed from arbitrary binary relations. Cattaneo [4] provided an axiomatization of approximation operators corresponding the ones produced by tolerance relations. The formulation is set in a more general context of bounded

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

posets. Some of axioms require the notions of open and closed definable elements. An inner approximation mapping associates an element to the largest open definable element from the bottom of the element, and an outer approximation mapping associates an element to the least closed definable elements from the top. This is an extension of Pawlak's definition as expressed by equation (1). Gehrke and Walker [15] and Iwinski [20] used similar methods. Furthermore, Cattaneo showed that in a quasi-BZ poset the sets of open and closed definable elements can be defined by using a pair of orthocomplementation mappings, which in turn can be defined axiomatically.

5 Rough Set Models and Uncertainty Measures

The relationships between the theory of belief functions and rough sets have been studied by many authors. Pawlak [43], Skowron [61, 62], Wong and Lingras [80], and Skowron and Grzymala-Busse [64] showed that one can derive a pair of belief and plausibility functions from a Pawlak rough set algebra. Harmanec, Klir, and Resconi [18] discussed interpretations of belief and plausibility functions in the framework of modal logic. They used the modal logic system T corresponding to a reflexive rough set algebra. Using a modal logic system corresponding to a reflexive and connected rough set algebra, Klir and Harmanec [22] discussed interpretations of necessity and possibility functions. Many researchers studied the connections of rough set models and modal logics [1, 10, 24, 30, 35, 36, 38, 40, 44, 45, 57, 58, 74, 76, 79]. Yao and Lingras [93] presented a detailed study on relationships between belief functions and rough set models. In this section, we review related results and present some new ones.

5.1 Belief and Plausibility Functions

In a rough set algebra $(2^U, \sim, \underline{apr}, \overline{apr}, \cap, \cup)$, the images of focal sets by the basic set assignment m produce the following partition of the universe:

$$M = \{m(A) \mid A \in F\}. \tag{33}$$

It is a basis of the σ-algebra $\sigma(M)$ generated by M. Let P denote a probability function on $\sigma(M)$. Suppose R is a binary relation corresponding to the approximation operators. The triple $apr = (U, R, (\sigma(M), P))$ is called a probabilistic approximation space.

An arbitrary subset $A \subseteq U$ is approximated by $\underline{apr}A$ and $\overline{apr}A$ in a probabilistic approximation space $apr = (U, R, (\sigma(M), P))$. They are measurable sets from $\sigma(M)$. We can therefore define a pair of lower and upper probabilities as follows:

$$\begin{aligned} \underline{P}(A) &= P(\underline{apr}A), \\ \overline{P}(A) &= P(\overline{apr}A). \end{aligned} \tag{34}$$

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

According to properties (L0)-(L3) and (U0)-(U3), $\underline{P}$ and $\overline{P}$ obey the axioms [44, 93]:

$$\begin{array}{ll} \text{(LP0)} & \underline{P}(A) = 1 - \overline{P}(\sim A), \\ \text{(LP1)} & \underline{P}(U) = 1, \\ \text{(LP2)} & \underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B) - \underline{P}(A \cup B), \\ \text{(UP0)} & \overline{P}(A) = 1 - \underline{P}(\sim A), \\ \text{(UP1)} & \overline{P}(\emptyset) = 0, \\ \text{(UP2)} & \overline{P}(A \cap B) \leq \overline{P}(A) + \overline{P}(B) - \overline{P}(A \cup B). \end{array}$$

Properties (LP0) and (UP0) state that $\underline{P}, \overline{P} : 2^U \longrightarrow [0,1]$ are dual functions [96]. Properties (LP2) and (UP2) can be expressed in a much stronger version: for every positive integer n and every collection $A_1, \ldots, A_n \subseteq U$,

$$\begin{aligned} \text{(LP2)} \quad & \underline{P}(A_1 \cup \ldots \cup A_n) \geq \sum_i \underline{P}(A_i) - \sum_{i<j} \underline{P}(A_i \cap A_j) \pm \\ & \ldots + (-1)^{n+1} \underline{P}(A_1 \cap \ldots \cap A_n), \\ \text{(UP2)} \quad & \overline{P}(A_1 \cap \ldots \cap A_n) \leq \sum_i \overline{P}(A_i) - \sum_{i<j} \overline{P}(A_i \cup A_j) \pm \\ & \ldots + (-1)^{n+1} \overline{P}(A_1 \cup \ldots \cup A_n). \end{aligned}$$

Properties with numbers 1 and 2 are subsets of axioms of belief and plausibility functions [60, 100]. They characterize generalized belief and plausibility functions proposed by Smets [73] under the open world assumption.

From the basic set assignment m, we define another function: for $A \subseteq U$,

$$bpa(A) = P(m(A)). \tag{35}$$

The function $bpa : 2^U \longrightarrow [0,1]$ is a basic probability assignment satisfying the axiom:

$$\text{(M1)} \qquad \sum_{A \subseteq U} bpa(A) = 1.$$

Axiom (M1) follows from axioms (A1) and (A2) of a basic set assignment. A subset A with $m(A) > 0$ is called a focal element. From the basic probability assignment, the pair of generalized belief and plausibility functions $\underline{P}$ and $\overline{P}$ are expressed as:

$$\begin{aligned} \underline{P}(A) &= P(\underline{apr}A) = P(\bigcup_{B \subseteq A} m(B)) = \sum_{B \subseteq A} P(m(B)) = \sum_{B \subseteq A} bpa(B), \\ \overline{P}(A) &= P(\overline{apr}A) = P(\bigcup_{B \cap A \neq \emptyset} m(B)) = \sum_{B \cap A \neq \emptyset} P(m(B)) = \sum_{B \cap A \neq \emptyset} bpa(B). \end{aligned} \tag{36}$$

This connection between a basic probability assignment and a pair of belief and plausibility functions has been studied extensively in the theory of belief functions [60, 73]. It should be pointed out that the preceding formulation is

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

an immediate generalization of the work by Skowron [63], and Skowron and Grzymala-Busse [64]. We used a probability function to define belief and plausibility values [93], instead of computing them from cardinalities of sets.

Consider a serial rough set model, Axioms (L5) and (U5) imply that $\underline{P}$ and $\overline{P}$ satisfy the axioms:

$$\begin{array}{ll} \text{(LP3)} & \underline{P}(\emptyset) = 0, \\ \text{(UP3)} & \overline{P}(U) = 1. \end{array}$$

It follows that $\underline{P}$ and $\overline{P}$ are a pair of belief and plausibility functions as originally proposed by Shafer [60]. The corresponding basic set assignment m satisfies axiom (A3), which implies that the basic probability assignment bpa obeys the axiom:

$$\text{(M2)} \qquad bpa(\emptyset) = 0.$$

From equation (36), one can easily see the correspondence between axiom (M2), (LP3) and (UP3).

A binary relation can be interpreted as a multivalued mapping from U to U using R_s. Our formulation of lower and upper probabilities is related to the framework suggested by Dempster [9]. In defining lower probability, Dempster used the following lower approximation:

$$\underline{apr}_d A = \{x \mid R_s(x) \neq \emptyset, R_s(x) \subseteq A\}. \tag{37}$$

Similar definition was also used by Wybraniec-Skardowska [83]. The lower approximation operator $\underline{apr}_d$ satisfies axioms (L2) and (L3). It is related to $\overline{apr}$ by:

$$\begin{array}{ll} \text{(LD0)} & \underline{apr}_d A = \overline{apr} U - \overline{apr} \sim A, \\ \text{(UD0)} & \overline{apr} A = \overline{apr} U - \underline{apr}_d \sim A. \end{array}$$

By definition, $\underline{apr}_d U = \overline{apr} U$. The corresponding lower and upper probabilies are defined by:

$$\begin{aligned} \underline{P}_d(A) &= \frac{P(\underline{apr}_d A)}{P(\overline{apr} U)}, \\ \overline{P}_d(A) &= \frac{P(\overline{apr} A)}{P(\overline{apr} U)}. \end{aligned} \tag{38}$$

They satisfy axioms (LP0)-(LP3) and (UP0)-(UP3), and hence are a pair of dual belief and plausibility functions. For a serial binary relation, our definition is the same as that of Dempster [9].

If the binary relation R is an equivalence relation, lower and upper probabilities are the same as rough probabilities introduced by Pawlak [44]. They have

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

been studied by many authors [44, 64, 80]. Particularly, instead of using a probability function, Grzymala-Busse [16], and Skowron and Grzymala-Busse [64] computed a pair of belief and plausibility functions as follows:

$$\underline{P}(A) = \frac{|\underline{apr}A|}{|U|},$$
$$\overline{P}(A) = \frac{|\overline{apr}A|}{|U|}, \tag{39}$$

where $|\cdot|$ denotes the cardinality of a set. This is equivalent to saying that one uses the following probability function defined on $\sigma(M)$: for $A \in \sigma(M)$,

$$P(A) = \frac{|A|}{|U|}. \tag{40}$$

Similar approach using cardinalities of sets was proposed by Harmanec, Klir, and Resconi [18] in a modal logic based framework.

5.2 Necessity and Possibility Functions

A binary relation R is connected if for every pair $x, y \in U$, xRy or yRx holds. Consider a *transitive* and *connected* rough set model. For any two elements $x, y \in U$, the connectiveness of R can be expressed by:

$$x \in R_s(y) \quad \text{or} \quad y \in R_s(x). \tag{41}$$

By transitivity, it implies:

$$R_s(x) \subseteq R_s(y) \quad \text{or} \quad R_s(y) \subseteq R_s(x). \tag{42}$$

That is, with respect to elements of U, the successor neighborhoods $R_s(x)$'s are nested subsets of U. It should be noted that transitivity and connectiveness are sufficient for the condition (42), they are not necessary as shown in the example of Section 3.4.

From definition $m(A) = \{x \mid R_s(x) = A\}$, we can conclude that if F_1 and F_2 are two distinct focal set of m, either $F_1 \subset F_2$ or $F_2 \subset F_1$ must hold. Thus, the set of all focal sets $F = \{F_1, \ldots, F_k\}$ is a family of nested subsets of U. Without loss of generality, we assume:

$$F_1 \subset \ldots \subset F_k. \tag{43}$$

With this constraint on the set of focal sets, from equations (25) and (26) we can conclude that approximation operators satisfy the axioms: for all $A, B \subseteq U$,

$$\text{(L6)} \qquad \underline{apr}A \subseteq \underline{apr}B \ \text{ or } \ \underline{apr}B \subseteq \underline{apr}A,$$
$$\text{(U6)} \qquad \overline{apr}A \subseteq \overline{apr}B \ \text{ or } \ \overline{apr}B \subseteq \overline{apr}A.$$

Under axioms (L2) and (U2), they can be equivalently expressed as:

$$\text{(L7)} \qquad \underline{apr}(A \cap B) = \underline{apr}A \ \text{ or } \ \underline{apr}(A \cap B) = \underline{apr}B,$$
$$\text{(U7)} \qquad \overline{apr}(A \cup B) = \overline{apr}A \ \text{ or } \ \overline{apr}(A \cup B) = \overline{apr}B.$$

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

These axioms have been examined by Yao *et al* [100] in a study of non-numeric functions, for which both approximation operators are examples.

The connectiveness of R implies that R is reflexive. Hence, property (T) and (T′) hold. A reflexive relation is a serial relation, which implies that property (D) holds, namely, properties (L5) and (U5) hold. This suggests that a transitive and connected a rough set model is a special serial model. Therefore, in a transitive and connected rough set model, the pair of lower and upper probabilities defined by:

$$\underline{P}(A) = P(\underline{apr}A), \\ \overline{P}(A) = P(\overline{apr}A), \tag{44}$$

is a pair of belief and plausibility functions. Moreover, from axioms (L7) and (U7), we have:

$$\text{(LP4)} \qquad \underline{P}(A \cap B) = \min(\underline{P}(A), \underline{P}(B)),$$
$$\text{(UP4)} \qquad \overline{P}(A \cup B) = \max(\overline{P}(A), \overline{P}(B)).$$

Axioms (LP4) and (UP4) imply axioms (LP2) and (UP2). Together with (LP1), (LP3), (UP1), and (UP3), they define a special class of belief and plausibility functions called consonant belief and plausibility functions. This class is studied independently in the theory of fuzzy sets under the name of necessity and possibility functions [11, 23].

From the family of nested focal sets, a family of pairwise disjoint subsets of U is defined by:

$$D_1 = F_1, \quad D_i = F_i - F_{i-1} \quad \text{for} \;\; i = 2, \ldots, k. \tag{45}$$

The upper approximation distribution $h(x) = \bigcup_{x \in A} m(A)$ satisfies the property: for $x_i \in D_i$, $i = 1, \ldots, k$,

$$h(x_k) \subset \ldots \subset h(x_1). \tag{46}$$

Using h, we define a function $\pi : U \longrightarrow [0, 1]$ as follows:

$$\pi(x) = P(h(x)) = P(\overline{apr}\{x\}) = \overline{P}(\{x\}). \tag{47}$$

According to axiom (U7), for a subset $A \subseteq U$, the value $\overline{P}(A)$ can be computed by:

$$\overline{P}(A) = \overline{P}(\bigcup_{x \in A} \{x\}) = \max_{x \in A} \overline{P}(\{x\}) = \max_{x \in A} \pi(x). \tag{48}$$

Thus, the function π is the corresponding possibility distribution of the possibility function $\overline{P}$. The arugments are valid if a special probability on $\sigma(M)$, as defined by equation (40), is used [22].

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

6 Rough Sets and Fuzzy Sets

The theory of rough sets has been shown to be related to fuzzy sets [82, 90], probabilistic modal logic [92], and Bayesian decision theory [95]. Based on these studies, this section examines the connection between fuzzy sets and rough sets using the notion of rough membership functions [48].

6.1 Rough Membership Functions

A fuzzy set $\mathcal{A}$ of U is defined by a membership function:

$$\mu_{\mathcal{A}} : U \longrightarrow [0, 1]. \tag{49}$$

There are many definitions for fuzzy set complement, intersection, and union. With the min-max system proposed by Zadeh [101], fuzzy set operators are defined component-wise as:

$$\begin{aligned} (\neg\mu_{\mathcal{A}})(x) &= 1 - \mu_{\mathcal{A}}(x), \\ (\mu_{\mathcal{A}} \sqcap \mu_{\mathcal{B}})(x) &= \min[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)], \\ (\mu_{\mathcal{A}} \sqcup \mu_{\mathcal{B}})(x) &= \max[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)]. \end{aligned} \tag{50}$$

Let $\mathcal{F}(U)$ denote the set of all fuzzy set defined on U. The system $(\mathcal{F}(U), \neg, \sqcap, \sqcup)$ may be regarded as a deviation of classical set algebra. In general, fuzzy set intersection and union may be defined by t-norms and t-conorms [23]. An important feature of such systems is that fuzzy sets operators are truth-functional. Membership functions of complement, intersection, and union can be computed solely from membership functions of the fuzzy sets involved.

Given a number $\alpha \in [0, 1]$, an α-cut of a fuzzy set is defined by:

$$\mathcal{A}_\alpha = \{x \in U \mid \mu_{\mathcal{A}}(x) \geq \alpha\}, \tag{51}$$

which is a subset of U. A strong α-cut is defined by:

$$\mathcal{A}_{\alpha^+} = \{x \in U \mid \mu_{\mathcal{A}}(x) > \alpha\}. \tag{52}$$

If α's are interpreted as thresholds, α-cuts are crisp set approximations of a fuzzy set at different levels.

In a Pawlak approximation space $apr = (U, E)$, the rough membership function of $A \subseteq U$ is defined by [48]:

$$\mu_A(x) = \frac{|A \cap [x]_E|}{|[x]_E|}. \tag{53}$$

Such a notion was used earlier by some authors for the development of probabilistic rough set models [49, 82]. One can easily see the similarity between rough membership functions and conditional probabilities. The rough membership value $\mu_A(x)$ may be interpreted as the probability that an arbitrary element of $[x]_E$ belongs to A. In fact, Yao and Wong [95] used conditional probabilities

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

$P(A \mid [x]_E)$ in an attempt to relate rough set theory to Bayesian decision theory. Yao and Lin [92] defined rough membership functions with respect to an arbitrary binary relation:

$$\mu_A(x) = \frac{|A \cap R_s(x)|}{|R_s(x)|}. \tag{54}$$

It is assumed that R is at least a serial relation. This notion of rough membership functions was introduced earlier by Pawlak [47] using an information function $I : U \longrightarrow 2^U$. The information function may be interpreted using the successor neighborhood operator R_s, i.e., $I(x) = R_s(x)$. In order to generalize such a notion further, we consider a probabilistic approximation space $apr = (U, R, (2^U, P))$. For $A \subseteq U$, its rough membership function is defined as:

$$\mu_A(x) = P(A \mid R_s(x)) = \frac{P(A \cap R_s(x))}{P(R_s(x))}. \tag{55}$$

Similarly, it is assumed that R is a serial relation and furthermore $P(R_s(x)) \neq 0$.

A rough set is defined by a rough membership function μ_A through a standard set $A \subseteq U$. Rough sets may be regarded as a special type of fuzzy sets. Let $R(U)$ denote the set of all rough sets defined on U. There are at most $|2^U|$ number of rough sets in $R(U)$, which is clearly a subset of $\mathcal{F}(U)$. There does not exist an one-to-one relationships between rough sets and subsets of U. Two distinct subsets of U may define the same rough set. Consequently, rough set operators cannot be defined directly using rough membership functions. Membership functions of rough sets corresponding to $\sim A$, $A \cap B$, and $A \cup B$ must be computed using set operators and the definition of rough membership functions:

$$\begin{aligned} (\neg \mu_A)(x) &= \mu_{\sim A}(x), \\ (\mu_A \sqcap \mu_B)(x) &= \mu_{A \cap B}(x), \\ (\mu_A \sqcup \mu_B)(x) &= \mu_{A \cup B}(x), \end{aligned} \tag{56}$$

where $\sim$, $\cap$, and $\cup$ are standard set-theoretic operators. Each rough set is obtained from a classical set, and rough set operators $\neg$, $\sqcap$, and $\sqcup$ are defined using classical set-theoretic operators. In other words, in formulating a set-oriented view of rough set theory we introduce notions of rough sets and rough set operators by modify the standard meanings of classical set and set-theoretic operators. The system $(R(U), \neg, \sqcap, \sqcup)$ is a deviation of classical set algebra $(2^U, \sim, \cap, \cup)$.

By laws of probability, intersection and union of rough sets are not truth-functional. We have:

(rm1) $\mu_{\sim A}(x) = 1 - \mu_A(x)$,

(rm2) $\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A \cap B}(x)$,

(rm3) $\max(0, \mu_A(x) + \mu_B(x) - 1) \leq \mu_{A \cap B}(x) \leq \min(\mu_A(x), \mu_B(x))$,

(rm4) $\max(\mu_A(x), \mu_B(x)) \leq \mu_{A \cup B}(x) \leq \min(1, \mu_A(x) + \mu_B(x))$.

Properties (rm1)-(rm4) clearly show that rough set theory provides a class of non-truth-functional fuzzy set systems [21, 90]. Such non-truth-functional systems need further attention.

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

By extending the notion of α-cuts of fuzzy sets, we define a pair of α-level lower and upper approximations with respect to a number $\alpha \in [0,1]$ as follows:

$$\begin{aligned} &\underline{apr}_\alpha A = \{x \mid \mu_A(x) \geq 1-\alpha\}, \\ &\overline{apr}_\alpha A = \{x \mid \mu_A(x) > \alpha\}. \end{aligned} \tag{57}$$

We call $\underline{apr}_\alpha$ and $\overline{apr}_\alpha$ probabilistic rough set approximation operators. They can be explained in the framework of Bayesian decision theory by minimizing certain loss function [99]. They have also been studied within the framework of variable precision rough set model proposed by Ziarko [104]. For probabilistic approximation operators, we have:

$$\begin{array}{ll} \text{(PL0)} & \underline{apr}_\alpha A = \sim\overline{apr}_\alpha \sim A, \\ \text{(PL1)} & \underline{apr}_\alpha U = U, \\ \text{(PL2)} & \underline{apr}_\alpha (A \cap B) \subseteq \underline{apr}_\alpha A \cap \underline{apr}_\alpha B, \\ \text{(PL3)} & \underline{apr}_\alpha (A \cup B) \supseteq \underline{apr}_\alpha A \cup \underline{apr}_\alpha B, \\ \text{(PU0)} & \overline{apr}_\alpha A = \sim \underline{apr}_\alpha \sim A, \\ \text{(PU2)} & \overline{apr}_\alpha \emptyset = \emptyset, \\ \text{(PU3)} & \overline{apr}_\alpha (A \cup B) \supseteq \overline{apr}_\alpha A \cup \overline{apr}_\alpha B, \\ \text{(PU4)} & \overline{apr}_\alpha (A \cap B) \subseteq \overline{apr}_\alpha A \cap \overline{apr}_\alpha B. \end{array}$$

As stated by (PL0) and (PU0), $\underline{apr}_\alpha$ and $\overline{apr}_\alpha$ are dual operators on 2^U. Properties (PL2) and (PU2) are much weaker versions of (L2) and (U2). We call the system $(2^U, \sim, \underline{apr}_\alpha, \overline{apr}_\alpha, \cap, \cup)$, $\alpha \in [0,1]$, a probabilistic rough set algebra, which is an extension of classical set algebra $(2^U, \sim, \cap, \cup)$. When the rough membership function is computed by equation (54), the algebraic and probabilistic approximations are related to each other by $\underline{apr}A = \underline{apr}_0 A$ and $\overline{apr}A = \overline{apr}_0 A$.

6.2 Interval Rough Membership Functions

When the membership values of elements of the universe are fuzzy sets, one obtains type-2 fuzzy sets [23]. A special class of type-2 fuzzy sets, called interval fuzzy sets, is defined by restricting membership values to subintervals of of $[0,1]$. Interval fuzzy sets are commonly known as Φ-fuzzy sets [50, 94]. In the theory of rough sets, we can introduce similar notions [90].

Based on the concepts of approximation operators and rough membership functions, we may define a pair of lower and upper rough membership functions:

$$\begin{aligned} \underline{\mu}_A(x) &= \mu_{\underline{apr}A}(x) = \frac{|\underline{apr}(A) \cap R_s(x)|}{|R_s(x)|}, \\ \overline{\mu}_A(x) &= \mu_{\overline{apr}A}(x) = \frac{|\overline{apr}(A) \cap R_s(x)|}{|R_s(x)|}, \end{aligned} \tag{58}$$

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

where it is assumed that R is at least a serial relation. By definition, we have: for $x \in U$,

$$
\begin{array}{ll}
\text{(ID)} & \underline{\mu}_A(x) \le \overline{\mu}_A(x), \\
\text{(IL0)} & \underline{\mu}_A(x) = 1 - \overline{\mu}_{\sim A}(x), \\
\text{(IL1)} & \underline{\mu}_U(x) = 1, \\
\text{(IL2)} & \underline{\mu}_{A\cup B}(x) \ge \underline{\mu}_A(x) + \underline{\mu}_B(x) - \underline{\mu}_{A\cap B}(x), \\
\text{(IL3)} & \underline{\mu}_{\emptyset}(x) = 0, \\
\text{(IU0)} & \overline{\mu}_A(x) = 1 - \underline{\mu}_{\sim A}(x), \\
\text{(IU1)} & \overline{\mu}_{\emptyset}(x) = 0, \\
\text{(IU2)} & \overline{\mu}_{A\cap B}(x) \le \overline{\mu}_A(x) + \overline{\mu}_B(x) - \overline{\mu}_{A\cup B}(x), \\
\text{(IU3)} & \overline{\mu}_U(x) = 1.
\end{array}
$$

According to axiom (ID), we call the pair of membership functions an interval rough membership function. Axioms (IL2) and (IU2) can be expressed in a much stronger form in parallel to axioms (LP2) and (UP3). This can be easily verified by using the arguments presented in Section 5 on uncertainty measures. While a rough membership function may be interpreted using a probability function, an interval rough membership function may be interpreted using a pair of belief and plausibility functions: for $A \subseteq U$,

$$
\begin{aligned}
Bel_x(A) &= \frac{|\underline{apr}(A) \cap R_s(x)|}{|R_s(x)|}, \\
Pl_x(A) &= \frac{|\overline{apr}(A) \cap R_s(x)|}{|R_s(x)|}.
\end{aligned} \tag{59}
$$

The results developed in the last few sections can be immediately applied to the study of interval rough membership functions.

If R is a reflexive binary relation, from axioms (T) and (T′) of approximation operators, we have a pair of additional axioms:

$$
\begin{array}{ll}
\text{(IT)} & \underline{\mu}_A(x) \le \mu_A(x), \\
\text{(IT}'\text{)} & \mu_A(x) \le \overline{\mu}_A(x).
\end{array}
$$

If R is a transitive and connected binary relation, the pair of belief and plausibility functions defined by equation (59) are necessity and possibility functions. In this case, we have:

$$
\begin{array}{ll}
\text{(IL4)} & \underline{\mu}_{A\cap B}(x) = \min(\underline{\mu}_A(x), \underline{\mu}_B(x)), \\
\text{(IU4)} & \overline{\mu}_{A\cup B}(x) = \max(\overline{\mu}_A(x), \overline{\mu}_A(x)).
\end{array}
$$

If R is an equivalence relation, the lower and upper rough membership functions are indeed the characteristic functions of $\underline{apr}A$ and $\overline{apr}A$.

The notion of interval rough membership functions can be extended further. One may use a probabilistic approximation space $apr = (U, R, (2^U, P))$. Rough and interval rough membership functions can be defined using equation (55). One may also use a pair of probabilistic approximation operators $\underline{apr}_\alpha$ and $\overline{apr}_\alpha$ in the definition interval rough membership functions.

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

7 Conclusion

The theory of rough sets is typically studied based on the notion of an approximation space and the induced lower and upper approximations of subsets of a universe. With the concepts introduced in this paper, such as successor neighborhoods, upper approximation distributions, and basic set assignments, one may obtain alternative representations of approximation operators. Moreover, relationships between the theory of rough sets and other theories of uncertainty can be established. A rough set model can be related to belief and plausibility functions. In this case, a basic set assignment corresponds to a basic probability assignment. If a rough set model is related to necessity and possibility functions, an upper approximation distribution corresponds to a possibility distribution. By studying these alternative representations, we may obtain more insights into theory of rough sets.

The relationships between rough sets and fuzzy sets can be established based on the concept of rough membership functions. Rough set theory provides a class of non-truth-functional fuzzy set systems, in which the membership values have a probabilistic interpretation. With lower and upper approximations of a set, one may define interval rough membership functions. They may be interpreted using belief and plausibility functions, or necessity and plausibility functions. The connection between these two non-classical set theories needs further exploration.

Generalized rough set models can be formulated and studied from both constructive and algebraic points of views. The constructive approach defines approximation operators based on a binary relation on the universe, while the algebraic approach characterizes approximation operators based on a set of axioms. The relationships between these two approaches can be easily established by showing the existence of a binary relation that produces the same approximation operators.

References

1. M. Banerjee and M.K. Chakraborty, Rough algebra, *Bulletin of the Polish Academy of Sciences, Mathematics*, **41**, 293-287, 1987.
2. M. Banerjee and M.K. Chakraborty, Rough consequence and rough algebra, in *Rough Sets, Fuzzy Sets and Knowledge Discovery*, W.P. Ziarko (Ed.), Springer-Verlag, London, 196-207, 1994.
3. G. Cattaneo, Generalized rough sets: preclusivity fuzzy-intuitionistic (BZ) lattices, *Studia Logica*, **58**, 47-77, 1997.
4. G. Cattaneo, Mathematical foundations of roughness and fuzziness, manuscript, Dipartimento di Scienze dell'Informazione, Univerita di Milano, via Comelico 39, I-20135 Milano, Italy, 1997. A preliminary version appeared in *Proceedings of the Fourth International Workshop on Rough Sets, Fuzy Sets, and Machine Discovery–RSFD'96*, 241-247, 1996.
5. S. Chanas and D. Kuchta, Further remarks on the relation between rough and fuzzy sets, *Fuzzy Sets and Systems*, **47**, 391-394, 1992.
6. B.F. Chellas, *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

7. S. Comer, An algebraic approach to the approximation of information, *Fundamenta Informaticae*, **14**, 492-502, 1991.
8. S. Comer, On connections between information systems, rough sets, and algebraic logic, in *Algebraic Methods in Logic and Computer Science*, Banach Center Publications 28, 117-124, 1993.
9. A.P. Dempster, Upper and lower probabilities induced by a multivalued mapping, *Annals of Mathematical Statistics*, **38**, 325-339, 1967.
10. D. Demri, Finite model property for a logic for data analysis, *Proceedings of the Sixth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 871-876, 1996.
11. D. Dubois, and P. Prade, *Possibility Theory: an Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.
12. D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, *International Journal of General Systems*, **17**, 191-209, 1990.
13. I. Düntsch, Rough sets and algebras, manuscript, School of Information and Software Engineering, Univerity of Ulster, Newtownabbey, BT 37 0QB, N. Ireland, 1996.
14. I. Düntsch and G. Gediga, The rough set model of data analysis – introduction and overview, manuscript, School of Information and Software Engineering, Univerity of Ulster, Newtownabbey, BT 37 0QB, N. Ireland, 1996.
15. Gehrke, M., and Walker, E., On the structure of rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, *40*, 235-245, 1992.
16. J.W. Grzymala-Busse, *Rough-set and Dempster-Shafer approaches to knowledge acquisition under uncertainty – a comparison*, unpublished manuscript, Department of Computer Science, University of Kansas, 1987.
17. S. Haack, *Philosophy of Logics*, Cambridge University Press, Cambridge, 1978.
18. D. Harmanec, J. Klir, and G. Resconi, On modal logic interpretation of Dempster-Shafer theory of evidence, *International Journal of Intelligent Systems*, *9*, 941-951, 1994.
19. T.B. Iwinski, Algebraic approach to rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **35**, 673-683, 1987.
20. T.B. Iwinski, Rough orders and rough concepts, *Bulletin of the Polish Academy of Sciences, Mathematics*, **36**, 187-192, 1988.
21. G.J. Klir, Multivalued logics versus modal logics: alternative frameworks for uncertainty modelling, in *Advances in Fuzzy Theory and Technology*, P.P. Wang, Ed., Department of Electrical Engineering, Duke University, Durham, North Carolina, 3-47, 1994.
22. G.J. Klir and D. Harmanec, On modal logic interpretation of possibility theory, *International Journal of Uncertainty, Fuzziness and Knowledge-based Systems*, **2**, 237-245, 1994.
23. G.J. Klir, and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, New Jersey, 1995.
24. B. Konikowska, A logic for reasoning about stochastic information, *Bulletin of the Polish Academy of Sciences, Technical Sciences*, **39**, 171-186, 1991.
25. B. Konikowska, A logic for reasoning about relative similarity, *Studia logica*, **58**, 185-226, 1997.
26. T.Y. Lin, Neighborhood systems: a qualitative theory for fuzzy and rough sets, *Proceedings 2nd Joint Conference on Information Sciences*, 255-258, 1995.
27. T.Y. Lin and N. Cercone (Eds.), *Rough Sets and Data Mining: Analysis for Imprecise Data*, Kluwer Academic Publishers, Boston, 411-425, 1997.

28. T.Y. Lin and Q. Liu, Rough approximate operators: axiomatic rough set theory, in *Rough Sets, Fuzzy Sets and Knowledge Discovery*, W.P. Ziarko (Ed.), Springer-Verlag, London, 256-260, 1994.
29. P.J. Lingras and S.K.M. Wong, Two perspectives of the Dempster-Shafer theory of belief functions, *International Journal of Man-machine Studies*, **33**, 467-848, 1990.
30. A. Nakamura, On a logic based on graded modalities, *IEICE Transactions on Information and Systems*, **E76-D**, 527-532, 1993.
31. A. Nakamura and J.M. Gao, A logic for fuzzy data analysis, *Fuzzy Sets and Systems*, **39**, 127-132, 1991.
32. A. Nakamura and J.M. Gao, On a KTB-modal fuzzy logic, *Fuzzy Sets and Systems*, **45**, 327-334, 1992.
33. H.T. Nguyen, Intervals in Boolean rings: approximation and logic, *Foundations of Computing and Decision Sciences*, **17**, 131-138, 1992.
34. H.T. Nguyen and E.A. Walker, A history and introduction to the algebra of conditional events and probability logic, *IEEE Transactions on Systems, Man, and Cybernetics*, **24**, 1671-1675, 1994.
35. E. Orlowska, Logic of indiscernibility relations, *Bulletin of the Polish Academy of Sciences, Mathematics*, **33**, 475-485, 1985.
36. E. Orlowska, Logical aspects of learning concepts, *International Journal of Approximate Reasoning*, **2**, 349-364, 1988.
37. E. Orlowska, Kripke semantics for knowledge representation logics, *Studia Logica*, **49**: 255-272, 1990.
38. E. Orlowska, Rough set semantics for non-classical logics, in *Rough Sets, Fuzzy Sets and Knowledge Discovery*, edited by W.P. Ziarko, Springer-Verlag, London, 143-148, 1994.
39. E.,Orlowska and Z. Pawlak, Measurement and indiscernibility, *Bulletin of Polish Academy of Sciences. Mathematics*, **32**, 617-624, 1984.
40. P. Pagliani, Rough sets theory and logic-algebraic structures, manuscript, Research Group on Knowledge and Communication Models, Via Imperia, 6. 00165 Roma, Italy, 1994.
41. P. Pagliani, Rough sets and Nelson algebras, *Fundamenta Informaticae*, **27**, 205-219, 1996.
42. Z. Pawlak, Information systems, theoretical foundations, *Information Systems*, **6**, 205-218, 1981.
43. Z. Pawlak, Rough sets. *International Journal of Computer and Information Sciences*, **11**, 341-356, 1982.
44. Z. Pawlak, Rough probability, *Bulletin of Polish Academy of Sciences. Mathematics*, **32**, 607-615, 1984.
45. Z. Pawlak, Rough logic, *Bulletin of the Polish Academy of Sciences, Technical Sciences*, **35**, 253-258, 1987.
46. Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, Boston, 1991.
47. Z. Pawlak, Hard and soft sets, in *Rough Sets, Fuzzy Sets and Knowledge Discovery*, W.P. Ziarko (Ed.), Springer-Verlag, London, 130-135, 1994.
48. Z. Pawlak and A. Skowron, Rough membership functions, in *Fuzzy Logic for the Management of Uncertainty*, L.A. Zadeh and J. Kacprzyk (Eds.), John Wiley & Sons, New York, 251-271, 1994.

49. Z. Pawlak, S.K.M. Wong, and W. Ziarko, Rough sets: probabilistic versus deterministic approach, *International Journal of Man-machine Studies*, **29**, 81-95, 1988.
50. W. Pedrycz, sets *Fuzzy Control and Fuzzy Systems*, second, extended, edition, John Wiley & Sons Inc., New York, 1993.
51. J.A. Pomykala, Approximation operations in approximation space, *Bulletin of Polish Academy of Sciences. Mathematics*, **35**, 653-662, 1987.
52. J.A. Pomykala, On definability in the nondeterministic information system, *Bulletin of the Polish Academy of Sciences, Mathematics* **36**, 193-210, 1988.
53. J. Pomykala and J.A. Pomykala, The Stone algebra of rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **36**, 495-508, 1988.
54. L.T. Polkowski, Mathematical morphology of rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **41**, 241-273, 1993.
55. L. Polkowski and A. Skowron, Rough mereology: a new paradigm for approximate reasoning, *International Journal of Approximate Reasoning*, **15**, 333-365, 1997.
56. M. Quafafou, α-RST: a generalization of rough set theory, manuscript, IRIN, University of Nates, 2 rue de la Houssininiere, BP 92208-44322, Nantes Cedex 03, France, 1997.
57. H. Rasiowa, On approximation logics: a survey, *Kurt Gödel Gesselschaft, Jahrbuch*, **1990**, 63-87, 1991.
58. H. Rasiowa, and A. Skowron, Rough concepts logic, in *Computation Theory*, *Lecture Notes in Computer Science*, No. 208, A. Skowron, (Ed.), Springer-Verlag, Berlin, 288-297, 1985.
59. H. Rasiowa, and A. Skowron, Approximation logic, in *Mathematical Methods of Specification and Synthesis of Software Systems'85*, *Lecture Notes in Computer Science*, No. 215, W. Bibel and K.P. Jantke (Eds.), Springer-Verlag, Berlin, 123-139, 1985.
60. G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, 1976.
61. A. Skowron, The relationship between the rough set theory and evidence theory, *Bulletin of Polish Academy of Sciences. Mathematics*, **37**, 87-90, 1989.
62. A. Skowron, The rough set theory and evidence theory, *Fundamenta Informaticae*, **13**, 245-262, 1990.
63. A. Skowron, Management of uncertainty in AI: a rough set approach, in *Incompleteness and Uncertainty in Information Systems*, V.S. Alagar, S. Bergler and F.Q. Dong (Eds.), Springer-Verlag, London, 69-86, 1994.
64. A. Skowron and J. Grzymala-Busse, From rough set theory to evidence theory, in *Advances in the Dempster-Shafer Theory of Evidence*, R.R. Yager, M. Fedrizzi, and J. Kacprzyk (Eds.), Wiley, New York, 193-236, 1994.
65. A. Skowron and C. Rauszer, The discernibility matrices and functions in information systems, in *Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory*, R. Slowinski (Ed.), Kluwer Academic Publishers, Boston, 331-362, 1992.
66. A. Skowron and J. Stepaniuk, Approximations of relations, in *Rough Sets, Fuzzy Sets and Knowledge Discovery*, W.P. Ziarko (Ed.), Springer-Verlag, London, 161-166, 1994.
67. A. Skowron and J. Stepaniuk, Generalized approximation spaces, in *Soft Computing*, T.Y. Lin and A.M. Wildberger (Eds.), The Society for Computer Simulation, San Diego, 18-21, 1995.

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

68. A. Skowron and J. Stepaniuk, Tolerance approximation spaces, *Fundamenta Informaticae*, **27**, 254-253, 1996.
69. R. Slowinski (Ed.), *Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory*, Kluwer Academic Publishers, Boston, 1992.
70. R. Slowinski, Strict and weak indiscernibility of objects described by quantitative attributes with overlapping norms, *Foundations of Computer and Decision Sciences*, **18**, 361-369, 1993.
71. R. Slowinski and J. Stefanowski, Rough-set reasoning about uncertain data, *Fundamenta Informaticae*, **27**, 229-243, 1996.
72. R. Slowinski and D. Vanderpooten, Similarity relation as a basis for rough approximations, in *Advances in Machine Intelligence & Soft-Computing*, P.P. Wang (Ed.), Department of Electrical Engineering, Duke University, Durham, North Carolina, 17-33, 1997.
73. P. Smets, Belief functions (with discussion), in *Non-Standard Logics for Automated Reasoning*, edited by P. Smets, A. Mamdani, D. Dubois, and H. Prade, Academic Press, New York, 282-285, 1988.
74. D. Vakarelov, A modal logic for similarity relations in Pawlak knowledge representation systems, *Fundamenta Informaticae*, **15**, 61-79, 1991.
75. D. Vakarelov, Modal logics for knowledge representation systems, *Theoretical computer science*, **90**, 433-456, 1991.
76. L. Vigneron and A. Wasilewska, Rough and modal algebras, *Proceedings of the Symposium on Modelling, Analysis and Simulation, Computational Engineering in Systems Applications*, 1107-1112, 1996.
77. A. Wasilewska, Topological rough algebras, in *Rough Sets and Data Mining: Analysis for Imprecise Data*, Lin, T.Y. and Cercone, N. (Eds.), Kluwer Academic Publishers, Boston, 411-425, 1997.
78. A. Wiweger, On topological rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **37**, 89-93, 1989.
79. S.K.M. Wong, A logical approach for modeling uncertainty, *Proceedings of the Sixth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 129-135, 1996.
80. S.K.M. Wong and P.J. Lingras, The compatibility view of Shafer-Dempster theory using the concept of rough set, *Methodologies of Intelligent Systems 4*, 33-42, 1989.
81. S.K.M. Wong, L.S. Wang, and Y.Y. Yao, On modeling uncertainty with interval structures, *Computational Intelligence*, **11**, 406-426, 1995.
82. S.K.M. Wong and W. Ziarko, Comparison of the probabilistic approximate classification and the fuzzy set model, *Fuzzy Sets and Systems*, **21**, 357-362, 1987.
83. U. Wybraniec-Skardowska, On a generalization of approximation space, *Bulletin of the Polish Academy of Sciences*, **37**, 51-61, 1989.
84. U. Wybraniec-Skardowska, Unit Operations, *ZeszytyNaukowe Wyzszej Szkoly Pedagogicznej im Powstancow Slaskich w Opolu, Matematyka*, **XXVII**, 113-129, 1992.
85. M. Wygralak, Rough sets and fuzzy sets – some remarks on interrelations, *Fuzzy Sets and Systems*, **29**, 241-243, 1989.
86. Y.Y. Yao, Two views of the theory of rough sets in finite universes, *International Journal of Approximate Reasoning*, **15**, 291-317, 1996.
87. Y.Y. Yao, Representation of rough set approximation operators, *Soft Computing in Intelligent Systems and Information Processing, Proceedings of 1996 Asian Fuzzy Systems Symposium*, Y.Y. Chen, K. Hirota, and J.Y. Yen (Eds.), IEEE Press, 7-12, 1996.

Yao, Y.Y., Generalized rough set models,
in: Rough Sets in Knowledge Discovery, Polkowski,
L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

88. Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, manuscript, Department of Computer Science, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1, 1997.
89. Y.Y. Yao, Constructive and algebraic approaches for generalized rough set models, *Bulletin of International Rough Set Society*, **1**, 22-29, 1997.
90. Y.Y. Yao, A comparative study of fuzzy sets and rough sets, manuscript, Department of Computer Science, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1, 1997.
91. Y.Y. Yao, Combination of rough and fuzzy sets based on α-level sets, in *Rough Sets and Data Mining: Analysis for Imprecise Data*, T.Y. Lin and N. Cercone (Eds.), Kluwer Academic Publishers, Boston, 301-321, 1997.
92. Y.Y. Yao and T.Y. Lin, Generalization of rough sets using modal logic, *Intelligent Automation and Soft Computing*, **2**, 103-120, 1996.
93. Y.Y. Yao and P. Lingras, Interpretations of belief functions in the theory of rough sets, to appear in *Information Sciences*, 1997.
94. Y.Y. Yao, and J. Wang, Interval based uncertain reasoning using fuzzy and rough sets, in *Advances in Machine Intelligence & Soft-Computing*, Volume IV, P.P. Wang (Ed.), Department of Electrical Engineering, Duke University, Durham, North Carolina, USA, 196-215, 1997.
95. Y.Y. Yao and S.K.M. Wong, A decision theoretic framework for approximating concepts, *International Journal of Man-machine Studies*, **37**, 793-809, 1992.
96. Y.Y. Yao and S.K.M. Wong, Representation, propagation and combination of uncertain information, *International Journal of General Systems*, **23**, 59-83, 1994.
97. Y.Y. Yao and S.K.M. Wong, Generalized probabilistic rough set models, *Soft Computing in Intelligent Systems and Information Processing, Proceedings of 1996 Asian Fuzzy Systems Symposium*, Y.Y. Chen, K. Hirota, and J.Y. Yen (Eds.), IEEE Press, 158-163, 1996.
98. Y.Y. Yao and S.K.M. Wong, Interval approaches for uncertain reasoning, to appear in Methodologies for Intelligent Systems – 10th International Symposium, ISMIS'97.
99. Y.Y. Yao, S.K.M. Wong, and T.Y. Lin, A review of rough set models, in *Rough Sets and Data Mining: Analysis for Imprecise Data*, Lin, T.Y. and Cercone, N. (Eds.), Kluwer Academic Publishers, Boston, 47-75, 1997.
100. Y.Y. Yao, S.K.M. Wong, and L.S. Wang, A non-numeric approach to uncertain reasoning, *International Journal of General Systems*, **23**, 343-359, 1995.
101. L.A. Zadeh, Fuzzy sets, Information and Control, **8**, 338-353, 1965.
102. W. Zakowski, On a concept of rough sets, *Demonstratio Mathematica* **XV**, 1129-1133, 1982.
103. W. Zakowski, Approximations in the space (U, Π), *Demonstratio Mathematica*, **XVI**, 761-769, 1983.
104. W. Ziarko, Variable precision rough set model, *Journal of Computer and System Sciences*, **46**, 39-59, 1993.
105. W.P. Ziarko (Ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery*, Springer-Verlag, London, 1994.

Yao, Y.Y., Generalized rough set models,
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L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, pp. 286-318, 1998.

$$
\begin{aligned}
R_s(x) &= \{y \mid x \in h(y)\} \\
&= A, x \in m(A) \\
&= \{y \mid x \notin \mathbf{L} \sim\{y\}\} \\
&= \{y \mid x \in \mathbf{H}\{y\}\}; \\
h(x) &= \{y \mid x \in R_s(y)\} \\
&= \bigcup_{x \in A} m(A) \\
&= \sim \mathbf{L} \sim \{x\} \\
&= \mathbf{H}\{x\}; \\
m(A) &= \{x \mid R_s(x) = A\} \\
&= \{x \mid \{y \mid x \in h(y)\} = A\} \\
&= \mathbf{L}A - \bigcup_{B \subset A} \mathbf{L}B \\
&= [\bigcap_{B \subset A} \mathbf{H} \sim B] - \mathbf{H} \sim A; \\
\mathbf{L}A &= \{x \mid R_s(x) \subseteq A\} \\
&= \sim \bigcup_{x \notin A} h(x) \\
&= \bigcup_{B \subseteq A} m(B) \\
&= \sim \mathbf{H} \sim A; \\
\mathbf{H}A &= \{x \mid R_s(x) \cap A \neq \emptyset\} \\
&= \bigcup_{x \in A} h(x) \\
&= \bigcup_{B \cap A \neq \emptyset} m(B) \\
&= \sim \mathbf{L} \sim A.
\end{aligned}
$$

Table 1. Relationships between representations of approximation operators

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serial	$\forall x \in U \exists y \in U[xRy]$,
	$\forall x \in U[R_s(x) \neq \emptyset]$,
(h1)	$\forall x \in U \exists y \in U[x \in h(y)]$,
	$\bigcup_{x \in U} h(x) = U$;
reflexive	$\forall x \in U[xRx]$,
	$\forall x \in U[x \in R_s(x)]$,
(h2)	$\forall x \in U[x \in h(x)]$;
symmetric	$\forall x, y \in U[xRy \Longrightarrow yRx]$,
	$\forall x, y \in U[x \in R_s(y) \Longrightarrow y \in R_s(x)]$,
(h3)	$\forall x, y \in U[x \in h(y) \Longrightarrow y \in h(x)]$;
transitive	$\forall x, y, z \in U[(xRy, yRz) \Longrightarrow xRz]$,
	$\forall x, y, z \in U[(y \in R_s(x), z \in R_s(y)) \Longrightarrow z \in R_s(x)]$,
	$\forall x, y \in U[y \in R_s(x) \Longrightarrow R_s(y) \subseteq R_s(x)]$,
(h4)	$\forall x, y, z \in U[x \in h(y), y \in h(z)) \Longrightarrow x \in h(z)]$,
	$\forall x, y \in U[x \in h(y) \Longrightarrow h(x) \subseteq h(y)]$;
Euclidean	$\forall x, y, z \in U[(xRy, xRz) \Longrightarrow yRz]$,
	$\forall x, y, z \in U[(y \in R_s(x), z \in R_s(x)) \Longrightarrow z \in R_s(y)]$,
	$\forall x, y \in U[y \in R_s(x) \Longrightarrow R_s(x) \subseteq R_s(y)]$,
(h5)	$\forall x, y, z \in U[(x \in h(y), x \in h(z)) \Longrightarrow y \in h(z)]$.

Table 2. Properties of binary relation and upper approximation distribution (Note that some of the properties are stated in several different forms.)

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relation	m	$\mathbf{L}$ and $\mathbf{H}$
any	(A1) $A \neq B \Longrightarrow m(A) \cap m(B) = \emptyset$ (A2) $\bigcup_{A \subseteq U} m(A) = U$	(L1) $\mathbf{L}U = U$ (L2) $\mathbf{L}(A \cap B) = \mathbf{L}A \cap \mathbf{L}B$ (U1) $\mathbf{H}\emptyset = \emptyset$ (U2) $\mathbf{H}(A \cup B) = \mathbf{H}A \cup \mathbf{H}B$
serial	(A3) $m(\emptyset) = \emptyset$	(L5) $\mathbf{L}\emptyset = \emptyset$ (U5) $\mathbf{H}U = U$
reflexive	(A4) $m(A) \subseteq A$	(T) $\mathbf{L}A \subseteq A$ (T′) $A \subseteq \mathbf{H}A$
symmetric	(A5) $[x \in m(A), y \in m(B)] \Longrightarrow$ $(y \in A \Longrightarrow x \in B)$	(B) $A \subseteq \mathbf{LH}A$ (B′) $\mathbf{HL}A \subseteq A$,
transitive	(A6) $[x \in m(A), y \in m(B)] \Longrightarrow$ $(y \in A \Longrightarrow B \subseteq A)$	(4) $\mathbf{L}A \subseteq \mathbf{LL}A$ (4′) $\mathbf{HH}A \subseteq \mathbf{H}A$
Euclidean	(A7) $[x \in m(A), y \in m(B)] \Longrightarrow$ $(y \in A \Longrightarrow A \subseteq B)$	(5) $\mathbf{H}A \subseteq \mathbf{LH}A$ (5′) $\mathbf{HL}A \subseteq \mathbf{L}X$

Table 3. Properties of basic set assignment and approximation operators