第 1 章

On Unifying Formal Concept Analysis and Rough Set Analysis

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1.1 Introduction

Formal concept analysis (FCA) and rough set analysis (RSA) are two complementary approaches for analyzing data[1,13,27,34]. Many proposals have been made to compare and combine the two theories, and to apply results from one theory to the other. Yao and Chen recently surveyed the existing studies and classified them into three groups[37]. The main results are briefly summarized below:

Comparative studies. The objectives are to gain an in-depth understanding of each theory and to investigate the similarities and differences of the two theories. Through such studies, one may obtain a more general data analysis framework. Kent examined the correspondence between similar notions used in both theories, and argued that they are in fact parallel to each other in terms of basic notions, issues and methodologies[7]. Ho developed a method of acquiring rough concepts[5], and Wu, Liu and Li proposed an approach for computing accuracies of rough concepts and studied the relationships between the indiscernibility relations and accuracies of rough concepts[31]. Pagliani used a Heyting algebra structure to connect concept lattices and approximation spaces together[11]. Wasilewski demonstrated that formal contexts and general approximation spaces can be mutually represented[26]. Qi, Wei and Li argued that the two theories have much in common in terms of the goals and methodologies[18]. Wolski investigated Galois connections in formal concept analysis and their relations to rough set analysis[28]. Yao compared the two theories based on the notion of definability, and showed that they deal with two different types of definability[34].

Applications of results from FCA to RSA. Some of the important results from FCA are the description of concepts through modal-style operators and hierarchical organization of concepts (i.e., concept lattices). One can apply those results into RSA. For example, one can construct additional concept lattices by considering more modal-style operators. Those lattices, their properties, and connections to the original concept lattice are studied extensively by Düntsch and Gediga[4], Wolski[28], and Yao[34,35].

Applications of results from RSA to FCA. Approximations, dependence analysis, attribute reduction, and object reduction are some of the main results from RSA. Many studies considered rough set approximations in formal concept lattice[6,12,19,20,22,28]. For example, Saquer and Deogun[19,21], Hu *et al.*[6], Düntsch and Gediga[2], Wolski[28], Pei and Xu[15], Yao and Chen[36], Pagliani and Chakraborty[12], and Shao and Zhang[22] proposed and examined various approximation operators in relation to FCA. Dependency analysis, attribute reduction and object reduction have recently received much attention. For example, Zhang, Wei and Qi[39], Li, Zhang and Wang[8], and Wu, Leung, Mi[32], studied the problem of reduction in FCA by drawing results from RSA. Ma, Zhang and Wang[9] investigated dependence space in FCA by using a similar technique of RSA.

From the existing studies, it is evident that there is a need for a unified theory. The main objective of this paper is to make a contribution to such a general theory. In particular, we examine the common underlying notion of FCA and RSA, namely, the definability of concepts, and the common process of constructing definable sets of objects and properties.

1.2 Definability of Concepts in a Binary Table

The classical view of concepts is explored in binary tables. Different types of definability are introduced.

1.2.1 Classical view of concepts

Concepts are the basic units of human thought and have been studied in many disciplines, including philosophy, psychology, education, computer science (in particular, machine learning), and many more[25]. There are many theoretical views of concepts, concept formation and learning[16,23,24,25]. The classical view treats concepts as entities with well-defined borderline and are describable by sets of singly necessary and jointly sufficient conditions[25]. Other views include the prototype view, the exemplar view, the frame view, and the theory view[25]. Each view captures specific aspects of concepts, and has a different implication for concept formation and learning. The applications of different views for inductive data analysis have been addressed by many authors[17,24,25].

We adopt the classical simple view of concepts. More complete treatments of the basic issues related to concept formation and organization can be found in the references[10,16,23,24,25].

In the classical view, every concept is understood as two parts, the intension and the extension of the concept[23,24,25]. The intension of a concept consists of all properties or attributes that are valid for all those objects to which the concept applies. The extension of a concept is the set of objects or entities which are instances of the concept. All objects in the extension have the same properties that characterize the concept. In other words, the intension of a concept is an abstract description of common features or properties shared by elements in the extension, and the extension consists of concrete examples of the concept. A concept is thus described jointly by its intension and extension.

1.2.2 Formal contexts

In order to derive a computational model for the classical view of concepts, we consider a simple representation scheme in which a finite set of objects are described by a finite set of attributes.

Let U be a set of objects and V a set of properties or attributes. The relationship between objects and properties can be formally defined by a binary relation R from U to V, $R \subseteq U \times V$. The triplet (U, V, R) is called a formal context in formal concept analysis[3,27], a binary information system in rough set theory[13,14], and a multi-valued mapping from U to V in the theory of evidence[21]. In this paper, we also refer to it as a binary table. It should be noted that any multi-valued table can be translated into a binary table using the scaling transformation in formal concept analysis[3].

For a pair $x \in U$ and $y \in V$, if xRy we say that object x has property y, or alternatively, property y is possessed by object x. For an object $x \in U$, its properties are given by the successors of x:

$$xR = \{y \in V \mid xRy\}. \tag{1.1}$$

For a property $y \in V$, the set of objects having y is given by the predecessors of y as follows:

$$Ry = \{x \in U \mid xRy\}. \tag{1.2}$$

We have two ways to extend the binary relation to subsets of objects and properties. For a subset $X \subseteq U$, we have:

$$\begin{aligned} X\underline{R} &= \{y \in V \mid \forall x \in X(xRy)\} \\ &= \bigcap_{x \in X} xR, \end{aligned} \tag{1.3}$$

$$\begin{aligned} X\overline{R} &= \{y \in V \mid \exists x \in X(xRy)\} \\ &= \bigcup_{x \in X} xR. \end{aligned} \tag{1.4}$$

Any object $x \in X$ has all the properties in $X\underline{R}$, and any property $y \in X\overline{R}$ is possessed by at least one object in X. Similarly, for a subset of properties, we have:

$$\begin{aligned} \underline{R}Y &= \{x \in U \mid \forall y \in Y(xRy)\} \\ &= \bigcap_{y \in Y} Ry, \end{aligned} \tag{1.5}$$

$$\begin{aligned} \overline{R}Y &= \{x \in U \mid \exists y \in Y(xRy)\} \\ &= \bigcup_{y \in Y} Ry. \end{aligned} \tag{1.6}$$

Any property $y \in Y$ is possessed by all the objects in $\underline{R}Y$, and any object in $\overline{R}Y$ has at least one property in Y.

1.2.3 Definability of concepts

The intension and the extension of a concept define and characterize each other. In formal concept analysis, intensions and extensions are expressed as subsets of properties and objects, respectively. In what follows, we also express intensions and extensions as logic formulas and formally introduce the notion of definability.

1.2.3.1 A logic language

In a binary table, one can introduce a logic language $\mathcal{L}$ to describe relationship between objects and properties. For simplicity, we use the same symbol R to denote both the binary relation and atomic formulas of $\mathcal{L}$. For any pair $(x, y) \in U \times V$, we have an atomic formula xRy. The formula xRy is true if x has property y, and false otherwise. Other formulas of $\mathcal{L}$ can be recursively defined. If f, f_1 and f_2 are formulas, then $\neg f$, $f_1 \wedge f_2$ and $f_1 \vee f_2$ are formulas. In this paper, we restrict the discussion to logic connectives $\wedge$ and $\vee$.

Consider now two special classes of formulas. For a subset of objects $X \subseteq U$ and a property $y \in V$, we have the conjunctive and disjunctive formulas:

$$\begin{aligned} &\bigwedge_{x \in X} xRy, \\ &\bigvee_{x \in X} xRy. \end{aligned} \tag{1.7}$$

They represent, respectively, that all objects in X have the property y and at least one object in X has the property y. For a subset of properties $Y \subseteq V$ and an object $x \in U$, we have:

$$\begin{aligned} &\bigwedge_{y \in Y} xRy, \\ &\bigvee_{y \in Y} xRy. \end{aligned} \tag{1.8}$$

They represent, respectively, that the object x has all properties in Y and the object x has at least one property in Y.

1.2.3.2 Mutual definability of sets of objects and properties

The intension and extension relationship may not hold between any pair of a subset of objects and a subset of properties. With the introduction of the logic language $\mathcal{L}$, we are able to consider definability more formally.

For a subset of objects $X \subseteq U$ and a subset of properties $Y \subseteq V$, we say that they are mutual conjunctively definable if the following conditions hold:

$$\text{(c1)} \qquad x \in X \Longleftrightarrow \bigwedge_{y \in Y} xRy,$$

$$\text{(c2)} \qquad \bigwedge_{x \in X} xRy \Longleftrightarrow y \in Y.$$

Condition (c1) states that one can find a formula based on the set of properties Y to define the set of objects X. Condition (c2), on the other hand, focuses on the same set of properties, but on the reverse relationship. It is important to note that the two conditions are not equivalent.

For a subset of objects $X \subseteq U$ and a subset of properties Y, we say that they are mutual disjunctively definable if the following conditions hold:

$$\text{(d1)} \qquad x \in X \Longleftrightarrow \bigvee_{y \in Y} xRy,$$

$$\text{(d2)} \qquad \bigvee_{x \in X} xRy \Longleftrightarrow y \in Y.$$

They are similar to (c1) and (c2), except that the disjunctive connective $\vee$ is used instead.

1.2.3.3 Definability of sets of objects

In many applications, mutual definability may be too strong. One may weaken it by considering only one of the two conditions.

For a set of objects $X \subseteq U$, we say that X is conjunctively definable if there is a set of properties $Y \subseteq V$ satisfying only the condition (c1):

$$\text{(c1)} \qquad x \in X \Longleftrightarrow \bigwedge_{y \in Y} xRy.$$

We say that X is disjunctively definable if there is a set of properties $Y \subseteq V$ satisfying only the condition (d1):

$$\text{(d1)} \qquad x \in X \Longleftrightarrow \bigvee_{y \in Y} xRy.$$

In those definitions, we only concentrate on the definability of a set of objects X, without considering the definability of the corresponding Y.

1.2.3.4 Definability of sets of properties

The definability of sets of properties is a mirror image of the definability of sets of objects.

For a set of properties $Y \subseteq V$, we say that Y is conjunctively definable if there is a set of objects $X \subseteq U$ satisfying only the condition (c2):

$$\text{(c2)} \qquad \bigwedge_{x \in X} xRy \Longleftrightarrow y \in Y.$$

We say that Y is disjunctively definable if there is a set of objects $X \subseteq U$ satisfying only the condition (d2):

$$\text{(d2)} \qquad \bigvee_{x \in X} xRy \Longleftrightarrow y \in Y.$$

The attention is turned to sets of properties.

1.2.4 Summary remarks

The model for the study of concepts in the context of binary tables is perhaps an over-simplified one. Our main purpose is to investigate the potential implications of such a simple model. The ideas derived are generally applicable.

The introduction of the definability of concept is only based on semantics considerations, which does not directly offer algorithms for the construction of the required sets of objects and properties. As will be shown in the next two sections, formal concept analysis and rough set analysis in fact offer such algorithms. Both theories only differ in the ways in which the notion of definability is interpreted. This allows us to derive a unifying framework and new results.

Data analysis using the two theories is based on set-theoretic operators, also called modal-style operators. They have been studied by many authors[2,4,29,30]. Formal concept analysis uses a pair of the sufficiency and the dual sufficiency operators, rough set analysis uses a pair of the necessity and the possibility operators. Definable sets of objects and properties can be characterized by those modal-style operators.

1.3 Formal Concept Analysis

Based on the sufficiency and the dual sufficiency operators, it is possible to characterize the family of definable sets of objects and properties.

1.3.1 Sufficiency and dual sufficiency operators

For a formal context (U, V, R), we define a pair of set-theoretic operators ${}^*, {}^\# : 2^U \longrightarrow 2^V$, called the sufficiency and the dual sufficiency operators[2], as follows:

$$
\begin{aligned}
X^* &= \{y \in V \mid \forall x \in U (x \in X \Longrightarrow xRy)\} \\
&= \{y \in V \mid X \subseteq Ry\} \\
&= \bigcap_{x \in X} xR \\
&= X\underline{R}, \\
X^\# &= X^{c*c} \\
&= \{y \in V \mid \exists x \in U (x \in X^c \wedge \neg(xRy))\} \\
&= \{y \in V \mid \neg(X^c \subseteq Ry)\} \\
&= \{y \in V \mid X^c \cap (Ry)^c \neq \varnothing\} \\
&= \{y \in V \mid X \cup Ry \neq U\}. \qquad (1.9)
\end{aligned}
$$

For a set of objects $X \subseteq U$, they associate a pair of sets of properties X^* and $X^\#$. Similarly, for any subset of properties $Y \subseteq V$, we can associate a pair of sets of objects $Y^*, Y^\# \subseteq U$. For simplicity, the same symbols are used for both operators from 2^U to 2^V and from 2^V to 2^U. Their roles can be easily seen from the context.

The pair of mappings, $^*: 2^U \longrightarrow 2^V$ and $^*: 2^V \longrightarrow 2^U$, is also called the derivation operators in formal concept analysis[3]. It is in fact a Galois connection between 2^U and 2^V.

1.3.2 Conjunctive concept lattice

By the definition of operators $^*: 2^U \longrightarrow 2^V$ and $^*: 2^V \longrightarrow 2^U$, we have two rules:

$$\text{(r1).} \qquad \bigwedge_{x\in X} xRy \Longleftrightarrow y \in X^*,$$

$$\text{(r2).} \qquad x \in Y^* \Longleftrightarrow \bigwedge_{y\in Y} xRy.$$

Rule (r1) suggests that a property is in X^* if and only if all objects in X have the property. Rule (r2) suggests that an object is in Y^* if and only if it has all the properties in Y. They provide a basis for the construction of mutual conjunctively definable sets of objects and properties.

Consider a pair of a set of objects and a set of properties (X, Y) with $X = Y^*$ and $Y = X^*$. By rules (r1) and (r2), we have:

$$\bigwedge_{x\in X} xRy \Longleftrightarrow y \in Y, \tag{1.10}$$

$$x \in X \Longleftrightarrow \bigwedge_{y\in Y} xRy. \tag{1.11}$$

That is, the pair satisfies both (c1) and (c2). Standard formal concept analysis focuses on such mutually definable (objects, properties) pairs.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called a conjunctive concept if $X = Y^*$ and $Y = X^*$. The set of objects X is referred to as the extension of the concept, and the set of properties is referred to as the intension of the concept. The set of all conjunctive concepts forms a complete lattice called a conjunctive concept lattice, or simply a concept lattice[3]. This lattice is denoted by $CL(U, V, R)$, where the first letter indicates that definability is given in term of conjunction. The meet and join of the lattice are given by:

$$(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \cap X_2, (Y_1 \cup Y_2)^{**}),$$
$$(X_1, Y_1) \vee (X_2, Y_2) = ((X_1 \cup X_2)^{**}, Y_1 \cap Y_2). \quad (1.12)$$

For any subset X of U, we have a conjunctive concept (X^{**}, X^*), and for any subset Y of V, we have a conjunctive concept (Y^*, Y^{**}).

The order relation in the concept lattice naturally defines the sub-concept relationship. For two conjunctive concepts (X_1, Y_1) and (X_2, Y_2), (X_1, Y_1) is a sub-concept of (X_2, Y_2) and (X_2, Y_2) is a sup-concept of (X_1, Y_1), written $(X_1, Y_1) \preceq (X_2, Y_2)$, if and only if $X_1 \subseteq X_2$, or equivalently, if and only if $Y_2 \subseteq Y_1$. A more general concept has a larger extension, and equivalently a smaller intension, than a more specific concept.

1.3.3 The family of object-oriented conjunctive concepts

By the definition of the sufficiency and the dual sufficiency operators, we can have:

$$\text{(r3).} \qquad x \in X \Longrightarrow \bigwedge_{y \in X^*} xRy,$$
$$\text{(r4).} \qquad x \in Y^{\#c} \Longleftrightarrow \bigwedge_{y \in Y^c} xRy.$$

For a pair (X, Y), $X \subseteq U, Y \subseteq V$, with $X = Y^{\#c}$ and $Y = X^{*c}$, it follows:

$$x \in X \Longleftrightarrow \bigwedge_{y \in Y} xRy. \quad (1.13)$$

That is, the pair only satisfies (c1). The set of properties Y provides a formula $\bigwedge_{y \in Y} xRy$ for defining the set of objects X. The reverse is not necessarily true.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called an object-oriented conjunctive concept if $X = Y^{\#c}$ and $Y = X^{*c}$, with extension X and intension Y. For any subset X of U, we have an object-oriented conjunctive concept

(X^{**}, X^{*c}), and for any subset Y of V, we have an object-oriented conjunctive concept $(Y^{\#c}, Y^{\#\#})$. It is interesting to examine the structure of the family of all object-oriented conjunctive concepts. We conjecture that it also is a lattice.

1.3.4 The family of property-oriented conjunctive concepts

Following the same argument, for any set of objects and any set of properties, the following rules can be obtained:

$$\text{(r5).} \qquad \bigwedge_{x \in Y^*} xRy \Longleftarrow y \in Y,$$
$$\text{(r6).} \qquad \bigwedge_{x \in X^c} xRy \Longleftrightarrow y \in X^{\#c}.$$

For a pair (X, Y), $X \subseteq U, Y \subseteq V$, with $X = Y^{*c}$ and $Y = X^{\#c}$, we immediately have:

$$\bigwedge_{x \in X} xRy \Longleftrightarrow y \in Y. \tag{1.14}$$

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called a property-oriented conjunctive concept if $X = Y^{*c}$ and $Y = X^{\#c}$, with extension X and intension Y. For any subset X of U, we have a property-oriented conjunctive concept $(X^{\#\#}, X^{\#c})$, and for any subset Y of V, we have a property-oriented conjunctive concept (Y^{*c}, Y^{**}). Again, we conjecture that the family of all property-oriented conjunctive concepts forms a lattice.

1.4 Rough Set Analysis

A fundamental notion of rough set analysis is approximation operators. We consider a type of approximations operators defined over two universes[15,22,29,30,38].

1.4.1 Necessity and possibility operators

With respect to a formal context (U, V, R), we can define another pair of dual set-theoretic operators: ${}^{\Box},{}^{\Diamond}: 2^U \longrightarrow 2^V$, called the lower and the upper approximation operators[29,30,33,38], as follows:

$$\begin{aligned} X^{\Box} &= \{y \in V \mid \forall x \in U(xRy \Longrightarrow x \in X)\} \\ &= \{y \in V \mid Ry \subseteq X\}, \qquad (1.15) \\ X^{\Diamond} &= \{y \in V \mid \exists x \in U(xRy \wedge x \in X)\} \\ &= \{y \in V \mid Ry \cap X \neq \varnothing\} \\ &= \bigcup_{x \in X} xR \\ &= X\overline{R}. \qquad (1.16) \end{aligned}$$

For a set of objects $X \subseteq U$, they associate a pair of sets of properties $X^{\Box}$ and $X^{\Diamond}$. Similarly, we can associate a pair of sets of objects with a set of properties $Y \subseteq V$ through a pair of set-theoretic operators, ${}^{\Box},{}^{\Diamond}: 2^V \longrightarrow 2^U$. Following our convention, the same symbols are again used for operators defined for both directions.

The approximation operators are also referred to as the necessity and the possibility operators[2].

1.4.2 The family of disjunctive concepts

By the definition of operators ${}^{\Diamond}: 2^U \longrightarrow 2^V$ and ${}^{\Diamond}: 2^V \longrightarrow 2^U$, we have a pair of rules:

$$\text{(r7).} \qquad \bigvee_{x \in X} xRy \Longleftrightarrow y \in X^{\Diamond},$$

$$\text{(r8).} \qquad x \in Y^{\Diamond} \Longleftrightarrow \bigvee_{y \in Y} xRy.$$

Rule (r7) states that a property is in $X^{\Diamond}$ if and only if at least one of the objects in X has the property. Similarly, rule (r8) states that an object is in $Y^{\Diamond}$ if and only if it has at least one property in Y. Compared with rules (r1) and (r2) of formal concept analysis, the definability in rough set analysis is characterized by disjunction, instead of conjunction.

Consider a pair of a set of objects and a set of properties (X, Y) with $X = Y^{\diamond}$ and $Y = X^{\diamond}$. By rules (6) and (7), we have:

$$\bigvee_{x \in X} xRy \Longleftrightarrow y \in Y, \tag{1.17}$$

$$x \in X \Longleftrightarrow \bigvee_{y \in Y} xRy. \tag{1.18}$$

That is, the pair satisfies both (d1) and (d2). This means that the set of objects X and the set of properties Y mutually determine each other. Rough set analysis studies such mutually definable (objects, properties) pairs.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called a disjunctive concept if $X = Y^{\diamond}$ and $Y = X^{\diamond}$. The set of objects X is referred to as the extension of the concept, and the set of properties is referred to as the intension of the concept.

We conjecture that the set of all disjunctive concepts forms a lattice.

1.4.3 Object-oriented disjunctive concept lattice

By the definition of approximation operators, we have a pair of rules:

$$\text{(r9).} \qquad x \in Y^{\diamond} \Longleftrightarrow \bigvee_{y \in Y} xRy,$$

$$\text{(r10).} \qquad x \in X \Longleftarrow \bigvee_{y \in X^{\Box}} xRy.$$

According to these rules, for a pair (X, Y) with $X \subseteq U, Y \subseteq V$, $X = Y^{\diamond}$ and $Y = X^{\Box}$, we have:

$$x \in X \Longleftrightarrow \bigvee_{y \in Y} xRy. \tag{1.19}$$

In this case, the pair of sets only satisfies (d1). That is, the set of objects X can be defined by the formula $\bigvee_{y \in Y} xRy$ in terms of properties in Y.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called an object-oriented disjunctive concept if $X = Y^{\diamond}$ and $Y = X^{\Box}$. If an object has a property in Y

then the object belongs to X. Furthermore, only objects in X have properties in Y. The set of objects X is called the extension of the concept (X, Y), and the set of the properties Y is called the intension.

The family of all object-oriented disjunctive concepts forms a lattice. The meet $\wedge$ and the join $\vee$ are defined by:

$$\begin{aligned}(X_1, Y_1) \wedge (X_2, Y_2) &= ((Y_1 \cap Y_2)^{\diamond}, Y_1 \cap Y_2) \\ &= ((X_1 \cap X_2)^{\Box\diamond}, Y_1 \cap Y_2), \\ (X_1, Y_1) \vee (X_2, Y_2) &= (X_1 \cup X_2, (X_1 \cup X_2)^{\Box}) \\ &= (X_1 \cup X_2, (Y_1 \cup Y_2)^{\diamond\Box}).\end{aligned}$$

For a set of objects $X \subseteq U$, we have an object-oriented disjunctive concept $(X^{\Box\diamond}, X^{\Box})$. For a set of properties $Y \subseteq V$, we have an object-oriented disjunctive concept $(Y^{\diamond}, Y^{\diamond\Box})$.

1.4.4 Property-oriented disjunctive concept lattice

For property-oriented disjunctive concepts, we consider a pair of rules defined by the approximation operators:

$$\begin{aligned}&(\text{r11}). && \bigvee_{x \in X} xRy \Longleftrightarrow y \in X^{\diamond}, \\ &(\text{r12}). && \bigvee_{x \in Y^{\Box}} xRy \Longrightarrow y \in Y.\end{aligned}$$

For a pair, (X, Y), $X \subseteq U, Y \subseteq V$, with $X = Y^{\Box}$ and $Y = X^{\diamond}$, we have:

$$\bigvee_{x \in X} xRy \Longleftrightarrow y \in Y. \tag{1.20}$$

That is, only (d2) holds for the pair of sets. For the set of properties Y, the set of objects X defines Y through the formula $\bigvee_{x \in X}$.

A pair (X, Y), $X \subseteq U, Y \subseteq V$, is called a property-oriented disjunctive concept if $X = Y^{\Box}$ and $Y = X^{\diamond}$. If a property is possessed by an object in X then the property must be in Y. Furthermore, only properties Y are possessed by objects in X.

The family of all property-oriented disjunctive concepts forms a lattice with meet $\wedge$ and join $\vee$ defined by:

$$\begin{aligned}(X_1, Y_1) \wedge (X_2, Y_2) &= (X_1 \cap X_2, (X_1 \cap X_2)^{\Diamond}) \\ &= (X_1 \cap X_2, (Y_1 \cap Y_2)^{\Box\Diamond}), \\ (X_1, Y_1) \vee (X_2, Y_2) &= ((Y_1 \cup Y_2)^{\Box}, Y_1 \cup Y_2) \\ &= ((X_1 \cup X_2)^{\Diamond\Box}, Y_1 \cup Y_2). \end{aligned} \tag{1.21}$$

For a set of objects $X \subseteq U$, we can construct a property-oriented disjunctive concept $(X^{\Diamond\Box}, X^{\Diamond})$. For a set of properties $Y \subseteq V$, there is a property-oriented disjunctive concept $(Y^{\Box}, Y^{\Box\Diamond})$.

1.5 Conclusion

In spite of their differences, formal concept analysis and rough set analysis share the same underlying notion of definability. They differ from each other on interpretations of definability. While formal concept analysis considers conjunctively definable concepts, rough set analysis considers disjunctively definable concepts. In addition, the two theories construct definable sets through modal-style operators with the same procedure. It appears that a general theory unifying formal concept analysis and rough set analysis is possible.

Acknowledgments

I wish to express my deep gratitude to Professor Wen-Xiu Zhang for his inspiration, friendship, and advice. He will always be my teacher. I would like to thank many colleagues and students in Professor Zhang's group for giving me opportunities to exchange ideas with them in the past few years. I am grateful to Dr. Wei-Zhi Wu, Dr. Guo-Fang Qiu, and Ping Xu who helped the organization of the Workshop. I thank my students, Yan Zhao and Yaohua Chen, for valuable discussions on formal concept analysis and rough set analysis.

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