On Modeling Uncertainty with Interval Structures

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Abstract

In this paper, we introduce the notion of interval structures in an attempt to establish a unified framework for representing uncertain information. Two views are suggested for the interpretation of an interval structure. A typical example using the *compatibility view* is the rough-set model in which the lower and upper approximations form an interval structure. Incidence calculus adopts the *allocation view* in which an interval structure is defined by the tightest lower and upper incidence bounds. The relationship between interval structures and interval-based numeric belief and plausibility functions is also examined. As an application of the proposed model, an algorithm is developed for computing the tightest incidence bounds.

Keywords belief functions, incidence calculus, interval structure, knowledge representation, rough sets, uncertainty management.

1 Introduction

In many situations, we often find ourselves in a state of uncertainty. This might stem from a lack of knowledge, or from the incompleteness or unreliability of the information at our disposal. It is therefore important to choose an appropriate *structure* to represent such information.

One may use numeric functions or non-numeric structures to represent uncertainty (Bhatnagar and Kanal 1986). The best known numeric method for modeling uncertainty is perhaps the Bayesian approach using probability functions (Neapolitan 1990; Pearl 1988). More recently, interval-based approaches have been adopted for uncertainty management. For example, one may use a pair of belief and plausibility functions to describe uncertainty by specifying an *interval* within which lies the true probability (Dubois and Prade 1986; Halpern and Fagin 1992; Shafer 1976; Smets 1988). This approach allows the assignment of one's belief to a proposition without necessarily committing the remaining belief to its negation. Another approach is possibility theory in which the uncertainty of a proposition is bounded by its necessity and possibility values (Dubois and Prade 1988; Klir and Folger 1988). These interval-based numeric approaches have been used successfully in the design of approximate reasoning systems.

On the other hand, non-numeric or qualitative methods are particularly useful for modeling uncertainty when numeric information is not readily available (Bhatnagar and Kanal 1986; Fine 1973; Luzeaux 1991; Satoh 1989). Typical examples of interval-based non-numeric methods include the rough-set theory (Iwinski 1987; Pawlak 1982, 1984; Pomykala and Pomykala, 1987), incidence calculus (Bundy 1985, 1986), and interval-set algebra (Yao 1993). In the rough-set model, a concept is characterized by a pair of ordinary (crisp) sets called the lower and upper approximations. The lower approximation contains the elements definitely belonging to the concept, whereas the upper approximation contains the elements possibly belonging to the concept. In incidence calculus, one assigns a lower bound and an upper bound to the incidence of a proposition. A lower bound represents the set of situations in which the proposition is definitely true, and an upper bound represents the set of situations in which the proposition could be true. In interval-set algebra, an interval set is used to define the bounds within which lies the true but unknown set. In the theory of fuzzy sets, the core of a fuzzy set is defined by collecting all elements with total membership, while the support is defined by collecting all elements with non-zero membership (Klir and Folger 1988). That is, a fuzzy set is qualitatively defined in terms of two crisp sets. More recently, Yao and Wong (1992) studied the rough-set and fuzzy-set models within the Bayesian decision-theoretic framework. In this method, a set can be approximated by different levels of lower and upper bounds depending on the particular application. The notion of two-fold fuzzy sets is another example of interval-based methods (Dubois and Prade 1990; Farinas del Cerro and Prade 1986).

All the above methods use the notion of *interval* in spite of their apparent differences. This suggests that there may exist a common framework for these methods. The present study extends our preliminary investigation on this topic (Wong, Wang and Yao 1992a, 1992b). We introduce the notion of *interval structures* to represent vague or imprecise information. An interval structure is defined as a pair of mappings between two Boolean algebras. Both the numeric and non-numeric interval-based methods will be analyzed using this new representation of uncertainty. It will be shown that the lower and upper approximations in the rough-set model, and the tightest lower and upper bounds in incidence calculus are a special kind of interval structure. We will present two interpretations of an interval structure, the *compatibility view* and the *allocation view*. To demonstrate the usefulness of this unified approach, we suggest an algorithm for computing the tightest incidence bounds.

This paper is organized as follows. In Section 2, we first define the concepts of a Boolean algebra before introducing the notion of interval structures. In Section 3, we discuss two views for interpreting an interval structure, which provide a plausible unified framework for modeling uncertainty. In Section 4, we examine the relationship between an interval structure and a pair of belief and plausibility functions.

For clarity, all the proofs of the lemmas and theorems developed in this paper are given in the Appendix.

2 Interval Structures

This section first reviews the basic concepts of a Boolean algebra pertinent to our discussion and then introduces the notion of interval structures.

2.1 Boolean algebra

A partially ordered set (poset) is a pair $(\mathcal{A}, \preceq)$, where $\mathcal{A}$ is a non-empty set and $\preceq$ is a reflexive, transitive, and antisymmetric binary relation on $\mathcal{A}$. If $a \preceq b$ and $a \neq b$, we write $a \prec b$. An element 0 of a poset $(\mathcal{A}, \preceq)$ is called a universal lower bound (zero) if $0 \preceq a$ for all $a \in \mathcal{A}$. Similarly, an element 1 of $(\mathcal{A}, \preceq)$ is called a universal upper bound (unit) if $a \preceq 1$ for all $a \in \mathcal{A}$. In the following discussion, only finite $\mathcal{A}$ is considered.

Let a, b, and c be elements of a poset $(\mathcal{A}, \preceq)$. We say that c is a least upper bound or a join of a and b if $a \preceq c$ and $b \preceq c$, and there exists no other element d in $\mathcal{A}$ such that $a \preceq d \preceq c$ and $b \preceq d \preceq c$. Least upper bounds are unique if they exist. Greatest lower bounds or meets are defined similarly, and they are also unique if they exist. The least upper bound of a and b is denoted by $a \vee b$, and the greatest lower bound of a and b is denoted by $a \wedge b$, where $\vee$ and $\wedge$ are referred to as the join and meet operations, respectively.

A poset $(\mathcal{A}, \preceq)$, in which any two elements have a join and a meet, is called a lattice, written as $(\mathcal{A}, \vee, \wedge)$. If a and b are elements in a lattice $(\mathcal{A}, \vee, \wedge)$, and

$$a \wedge b = 0 \quad \text{and} \quad a \vee b = 1, \tag{1}$$

we say b is a complement of a. In general, an element in a lattice may have more than one complement. A complement of a is denoted by $\neg a$. A lattice is said to be complemented if every element has at least one complement. A lattice is distributive if for all elements a, b, and c in $(\mathcal{A}, \vee, \wedge)$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{2}$$

or equivalently,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c). \tag{3}$$

A Boolean algebra $(\mathcal{A}, \vee, \wedge, \neg, 0, 1)$ is a complemented distributive lattice with a zero 0 and a unit 1. In a Boolean algebra, complements are uniquely defined.

2.2 An interval structure over two Boolean algebras

For notational convenience, the same symbols will be used to denote the binary operations and the universal bounds in different Boolean algebras.

Given a Boolean algebra $(\mathcal{A}, \vee, \wedge, \neg, 0, 1)$, we call $\mathcal{E} \subseteq \mathcal{A}$ a *multiplicative* subset of $\mathcal{A}$ if $a \wedge b$ is in $\mathcal{E}$ whenever a and b are in $\mathcal{E}$, and an *additive* subset of $\mathcal{A}$ if $a \vee b$ is in $\mathcal{E}$ whenever a and b are in $\mathcal{E}$. Let $\mathcal{E}$ be a multiplicative subset of $\mathcal{A}$ containing both 0 and 1. Let $\mathcal{X}$ be a multiplicative subset of $\mathcal{B}$ in another Boolean algebra $(\mathcal{B}, \vee, \wedge, \neg, 0, 1)$, containing both 0 and 1. We call a mapping $g\colon\ \mathcal{E} \longrightarrow \mathcal{X}$ a $\wedge$-*homomorphism* if $g(0) = 0$, $g(1) = 1$, and $g(a \wedge b) = g(a) \wedge g(b)$ for all a, $b \in \mathcal{E}$. Let $\mathcal{E}$ be an additive subset of $\mathcal{A}$ containing both 0 and 1, and $\mathcal{X}$ be an additive subset of $\mathcal{B}$ containing both 0 and 1. We call a mapping $h\colon\ \mathcal{E} \longrightarrow \mathcal{X}$ a $\vee$-*homomorphism* if $h(0) = 0$, $h(1) = 1$, and $h(a \vee b) = h(a) \vee h(b)$ for all $a, b \in \mathcal{E}$. Obviously, the sets $\mathcal{A}$ and $\mathcal{B}$ are both multiplicative and additive.

Given any two arbitrary elements a and b of $\mathcal{A}$, $a \preceq b$ holds if and only if $a = a \wedge b$. Let $g\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ be a $\wedge$-*homomorphism.* Suppose $a \preceq b$. By definition, $g(a) = g(a \wedge b) = g(a) \wedge g(b)$. This implies that $g(a) \preceq g(b)$. Therefore, $a \preceq b \Longrightarrow g(a) \preceq g(b)$. From $a \preceq a \vee b$ and $b \preceq a \vee b$, we obtain $g(a) \preceq g(a \vee b)$ and $g(b) \preceq g(a \vee b)$. This means that $g(a) \vee g(b) \preceq g(a \vee b)$. Similarly, for a $\vee$-homomorphism $h:\ \mathcal{A} \longrightarrow \mathcal{B}$, we can show $a \preceq b \Longrightarrow h(a) \preceq h(b)$ and $h(a \wedge b) \preceq h(a) \wedge h(b)$. These two properties are stated in the following lemmas.

Lemma 1 *Suppose $g\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ is a $\wedge$-homomorphism, and $h\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ is a $\vee$-homomorphism. Then $a \preceq b \Longrightarrow g(a) \preceq g(b)$ and $a \preceq b \Longrightarrow h(a) \preceq h(b)$.*

Lemma 2 *If $g\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ is a $\wedge$-homomorphism, then $g(a) \vee g(b) \preceq g(a \vee b)$. If $h\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ is a $\vee$-homomorphism, then $h(a \wedge b) \preceq h(a) \wedge h(b)$.*

In this study, we are primarily interested in *dual* homomorphisms.

Definition 1 *Suppose $\underline{f}\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ and $\overline{f}\colon\ \mathcal{A} \longrightarrow \mathcal{B}$ are two mappings from a Boolean algebra $(\mathcal{A}, \vee, \wedge, \neg, 0, 1)$ to another Boolean algebra $(\mathcal{B}, \vee, \wedge, \neg, 0, 1)$. We say that $\underline{f}$ and $\overline{f}$ are dual mappings if $\overline{f}(a) = \neg \underline{f}(\neg a)$ for every $a \in \mathcal{A}$.*

The next two lemmas summarize some of the important properties of dual $\wedge$- and $\vee$-homomorphisms.

Lemma 3 *Given a $\wedge$-homomorphism $\underline{f}$, its dual mapping $\overline{f}$ obtained from $\overline{f}(a) = \neg \underline{f}(\neg a)$ is a $\vee$-homomorphism, and vice versa.*

Lemma 4 *Suppose $\underline{f}$ and $\overline{f}$ are a pair of dual $\wedge$- and $\vee$-homomorphisms. Then for any $a \in \mathcal{A}$,*

$$\underline{f}(a) \preceq \overline{f}(a). \tag{4}$$

Based on Lemma 4, a pair of dual $\wedge$- and $\vee$-homomorphisms characterizes an element $a \in \mathcal{A}$ in terms of two elements $\underline{f}(a), \overline{f}(a) \in \mathcal{B}$ with $\underline{f}(a) \preceq \overline{f}(a)$. This pair of elements can be interpreted as the two extreme points of an *interval* in the Boolean algebra $(\mathcal{B}, \vee, \wedge, \neg, 0, 1)$:

$$[\underline{f}(a), \overline{f}(a)] = \{x \mid x \in \mathcal{B} \text{ and } \underline{f}(a) \preceq x \preceq \overline{f}(a)\}. \tag{5}$$

We may call $\underline{f}(a)$ the lower bound of a, $\overline{f}(a)$ the upper bound of a, and $[\underline{f}(a), \overline{f}(a)]$ the interval representation of a. The intervals associated with different elements of $\mathcal{A}$ satisfy the following properties:

$$\begin{array}{ll}
\text{(L1)} & \underline{f}(a) \vee \underline{f}(b) \preceq \underline{f}(a \vee b), \\
\text{(L2)} & \underline{f}(a) \wedge \underline{f}(b) = \underline{f}(a \wedge b), \\
\text{(L3)} & \underline{f}(0) = 0, \\
\text{(L4)} & \underline{f}(1) = 1,
\end{array}$$

and

$$\begin{array}{ll}
\text{(U1)} & \overline{f}(a \vee b) = \overline{f}(a) \vee \overline{f}(b), \\
\text{(U2)} & \overline{f}(a \wedge b) \preceq \overline{f}(a) \wedge \overline{f}(b), \\
\text{(U3)} & \overline{f}(0) = 0, \\
\text{(U4)} & \overline{f}(1) = 1.
\end{array}$$

These properties suggest that the structure induced by a pair of dual $\wedge$- and $\vee$-homomorphisms needs more attention. It should be noted that the above properties are not necessarily independent. For example, Lemma 2 implies that (L2) $\Longrightarrow$ (L1) and (U1) $\Longrightarrow$ (U2).

Definition 2 *Let $(\mathcal{A}, \vee, \wedge, \neg, 0, 1)$ and $(\mathcal{B}, \vee, \wedge, \neg, 0, 1)$ be two Boolean algebras. Given two mappings $\underline{f}\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $\overline{f}\colon \mathcal{A} \longrightarrow \mathcal{B}$, we call the pair $(\underline{f}, \overline{f})$ an interval structure if $\underline{f}$ is a $\wedge$-homomorphism, $\overline{f}$ is a $\vee$-homomorphism, and $\overline{f}(a) = \neg\underline{f}(\neg a)$ for all $a \in \mathcal{A}$, i.e., $\underline{f}$ and $\overline{f}$ are a pair of dual $\wedge$- and $\vee$-homomorphisms.*

An alternate way of defining an interval structure is through another mapping $j\colon \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the axioms:

$$\begin{array}{ll}
\text{(A1)} & j(0) = 0, \\
\text{(A2)} & \bigvee_{a \in \mathcal{A}} j(a) = 1, \\
\text{(A3)} & a \neq b \Longrightarrow j(a) \wedge j(b) = 0.
\end{array}$$

This mapping is called a *basic assignment*, and an element $a \in \mathcal{A}$ with $j(a) \neq 0$ is called a *focal element*. From a given j, one can define a mapping $\underline{f}$: for all

$a \in \mathcal{A}$,

$$\underline{f}(a) = \bigvee_{b \preceq a} j(b), \tag{6}$$

and another mapping $\overline{f}$ by the relationship $\overline{f}(a) = \neg \underline{f}(\neg a)$. The mapping $\overline{f}$ can be equivalently defined by:

$$\overline{f}(a) = \bigvee_{a \wedge b \neq 0} j(b). \tag{7}$$

It can be easily verified that the pair $(\underline{f}, \overline{f})$ is an interval structure. Conversely, given an interval structure $(\underline{f}, \overline{f})$, we can construct the basic assignment j by the formula: for all $a \in \mathcal{A}$,

$$j(a) = \underline{f}(a) \wedge \neg(\bigvee_{b \prec a} \underline{f}(b)). \tag{8}$$

These observations are summarized in the following theorem and its corollary.

Theorem 1 *Let $\underline{f}$ and $\overline{f}$ be two mappings from a Boolean algebra $\mathcal{A}$ to another Boolean algebra $\mathcal{B}$ with $\overline{f}(a) = \neg \underline{f}(\neg a)$ for every $a \in \mathcal{A}$. The pair $(\underline{f}, \overline{f})$ is an interval structure, if and only if there exists a basic assignment $j: \mathcal{A} \longrightarrow \mathcal{B}$ such that for all $a \in \mathcal{A}$,*

$$\underline{f}(a) = \bigvee_{b \preceq a} j(b).$$

Corollary 1 *Suppose $(\underline{f}, \overline{f})$ is an interval structure defined by a basic assignment j: $\mathcal{A} \longrightarrow \mathcal{B}$. Then for all $a \in \mathcal{A}$,*

$$j(a) = \underline{f}(a) \wedge \neg(\bigvee_{b \prec a} \underline{f}(b)).$$

So far, we have only introduced the abstract notion of interval structures and discussed some of their properties. In subsequent sections, we will present two plausible interpretations of an interval structure, and study the relationship between interval structures and other schemes for representing uncertainty.

3 Interpretations of an Interval Structure

This section presents two interpretations of an interval structure, the *compatibility view* and the *allocation views*. These views were used by Shafer (1987) to interpret belief functions (Lingras and Wong 1990). The compatibility view is related to the notion of rough sets, whereas the allocation view is connected with incidence calculus. The following discussion shows that interval structures indeed provide a unified framework for representing a variety of uncertain information.

3.1 The compatibility view

The notion of rough sets was introduced by Pawlak (1982) for approximating a *concept* by using two ordinary sets referred to as the lower and upper approximations. This model has been used successfully for automatic classification and rule generation in machine learning (Pawlak 1984; Pawlak, Wong and Ziarko 1988). We will show that the compatibility view of an interval structure leads to the notion of *generalized* rough sets.

Suppose a set of *descriptions* $W = \{w_1, w_2, \ldots, w_m\}$ is used to characterize a given set of *objects* $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$. This relationship can be formally defined in terms of a compatibility relation $\mathcal{C}$ between the elements of Θ and W:

$$\theta\ \mathcal{C}\ w \Longleftrightarrow \text{ object } \theta \text{ is characterized by description } w. \tag{9}$$

Such a relation defines a knowledge representation system. Clearly, in this system it may not be possible to precisely characterize an arbitrary concept (represented by a subset $A \subseteq \Theta$) using the descriptions in W. However, one can approximately describe any concept A by constructing the following two mappings $\underline{\mathcal{C}}\colon\ 2^{\Theta} \longrightarrow 2^{W}$ and $\overline{\mathcal{C}}\colon\ 2^{\Theta} \longrightarrow 2^{W}$:

$$\underline{\mathcal{C}}(A) = \{w \in W \mid \Gamma_{\mathcal{C}}(w) \subseteq A\} = \bigcup_{\Gamma_{\mathcal{C}}(w) \subseteq A} \{w\} \tag{10}$$

and

$$\overline{\mathcal{C}}(A) = \{w \in W \mid \Gamma_{\mathcal{C}}(w) \cap A \neq \emptyset\} = \bigcup_{\Gamma_{\mathcal{C}}(w) \cap A \neq \emptyset} \{w\}, \tag{11}$$

where

$$\Gamma_{\mathcal{C}}(w) = \{\theta \in \Theta \mid \theta\ \mathcal{C}\ w\}. \tag{12}$$

The set $\underline{\mathcal{C}}(A)$, a subset of W, is the union of all the singleton subsets of W whose elements are compatible with only the elements of A, whereas $\overline{\mathcal{C}}(A)$ is the union of all the singleton subsets of W whose elements are compatible with at least one element of A. Obviously, $\underline{\mathcal{C}}(A) \subseteq \overline{\mathcal{C}}(A)$. These two sets can be interpreted as the *lower* and *upper descriptions* of concept A using the description of the objects in A. The interval $[\underline{\mathcal{C}}(A), \overline{\mathcal{C}}(A)]$ can thus be considered as a *generalized rough set* of A, which provides an approximate characterization of A by the description of its members. Furthermore, we obtain the deterministic decision rule, written $\underline{\mathcal{C}}(A) \rightarrow A$, which means that an object in Θ with a description in $\underline{\mathcal{C}}(A)$ *definitely* belongs to A. On the other hand, the non-deterministic rule, written $\overline{\mathcal{C}}(A) \leadsto A$, says that an object in Θ with a description in $\overline{\mathcal{C}}(A)$ *possibly* belongs to A (Wong, Yao and Wang 1993). It is important to note that the pair of mappings $(\underline{\mathcal{C}}, \overline{\mathcal{C}})$ forms an interval structure over the two Boolean algebras $(2^{\Theta}, \cup, \cap, \neg, \emptyset, \Theta)$ and $(2^{W}, \cup, \cap, \neg, \emptyset, W)$.

Now if one assumes that each object has a *unique* description, then $\mathcal{C}$ induces the following equivalence relation R on Θ:

$$\theta_i \; R \; \theta_j \Longleftrightarrow \text{ there exists a } w \in W \text{ such that } \theta_i \; \mathcal{C} \; w \text{ and } \theta_j \; \mathcal{C} \; w. \tag{13}$$

That is, if two elements θ_i and θ_j are characterized by the same description w, they are considered to be *equivalent*. The relation R partitions the set Θ into a family of disjoint subsets, $\{[w_1]_{\mathcal{C}}, [w_2]_{\mathcal{C}}, \ldots, [w_m]_{\mathcal{C}}\}$, where $[w]_{\mathcal{C}}$ denotes an equivalence class of R. For this special case, from equations (10) and (11), we immediately obtain:

$$\underline{R}(A) = \bigcup_{[w]_{\mathcal{C}} \subseteq A} [w]_{\mathcal{C}}, \tag{14}$$

$$\overline{R}(A) = \bigcup_{[w]_{\mathcal{C}} \cap A \neq \emptyset} [w]_{\mathcal{C}}, \tag{15}$$

which are in fact the original lower and upper approximations of A introduced by Pawlak (1982). The interval $[\underline{R}(A), \overline{R}(A)]$ is the so called *rough set* of A, which provides an approximate characterization of A by the objects that share the same description of its members. The pair of mappings $(\underline{R}, \overline{R})$ defined by equations (14) and (15) is an interval structure over the same Boolean algebra $(2^{\Theta}, \cup, \cap, \neg, \emptyset, \Theta)$.

Conversely, given an interval structure $(\underline{f}, \overline{f})$ over the two Boolean algebras $(2^{\Theta}, \cup, \cap, \neg, \emptyset, \Theta)$ and $(2^{W}, \cup, \cap, \neg, \emptyset, W)$, one can construct a compatibility relation $\mathcal{C}$ between Θ and W from the corresponding basic assignment $j\colon 2^{\Theta} \longrightarrow 2^{W}$. That is, for any focal element F (i.e., $j(F) \neq \emptyset$) and all $\theta \in F$, we can define $\theta \; \mathcal{C} \; w$ for every element $w \in j(F)$.

It is perhaps worth mentioning that we can always construct an interval structure over any two Boolean algebras $(\mathcal{A}, \vee, \wedge, \neg, 0, 1)$ and $(\mathcal{B}, \vee, \wedge, \neg, 0, 1)$ from a compatibility relation between $\mathcal{A}$ and $\mathcal{B}$. The two extreme points of an interval for any element $a \in \mathcal{A}$ can be obtained from equations (10) and (11) by replacing $\subseteq$ by $\preceq$, $\cup$ by $\vee$, $\cap$ by $\wedge$, and the elements of subsets by the minterms of the corresponding Boolean algebra.

The following example illustrates the main idea presented in this section.

Example 1 *Consider a set of objects* $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ *and a set of descriptions* $W = \{w_1, w_2, w_3\}$. *Suppose the compatibility relation* $\mathcal{C}$ *is:*

$$\theta_1 \; \mathcal{C} \; w_1, \qquad \theta_2 \; \mathcal{C} \; w_2, \qquad \theta_3 \; \mathcal{C} \; w_2, \qquad \theta_4 \; \mathcal{C} \; w_3,$$

which gives:

$$\Gamma_{\mathcal{C}}(w_1) = \{\theta_1\}, \quad \Gamma_{\mathcal{C}}(w_2) = \{\theta_2, \theta_3\}, \quad \Gamma_{\mathcal{C}}(w_3) = \{\theta_4\}.$$

Consider a concept $A = \{\theta_1, \theta_2\}$. *From equations (10) and (11), we obtain:*

$$\underline{\mathcal{C}}(A) = \{w_1\}, \qquad \overline{\mathcal{C}}(A) = \{w_1, w_2\}.$$

The interval $[\{w_1\}, \{w_1, w_2\}]$ *is the generalized rough set of* A*. The deterministic and non-deterministic rules for* A *are:*

$$\{w_1\} \rightarrow A, \qquad \{w_1, w_2\} \rightsquigarrow A.$$

The equivalence classes induced by $\mathcal{C}$ *are:* $[w_1]_{\mathcal{C}} = \{\theta_1\}$*,* $[w_2]_{\mathcal{C}} = \{\theta_2, \theta_3\}$ *and* $i[w_3]_{\mathcal{C}} = \{\theta_4\}$*. According to equations (14) and (15), the lower and upper approximations of* A *are:*

$$\underline{R}(A) = \{\theta_1\}, \qquad \overline{R}(A) = \{\theta_1, \theta_2, \theta_3\}.$$

By definition, the interval $[\{\theta_1\}, \{\theta_1, \theta_2, \theta_3\}]$ *is the rough set of* $\{\theta_1, \theta_2\}$*.*

The concepts of two-fold fuzzy sets, the core, the support and α-cuts of fuzzy sets, and interval sets are closely related to rough sets (Wong, Wang and Yao 1992a). Their relationships to interval structures can be established as well. It should be noted that in this paper the notion of interval structures is defined using the mathematical structure called Boolean algebra. In general, one may introduce similar notions based on other mathematical structures, in the same way that the rough-set theory is developed. It will be interesting and worthwhile to extend the present study by using mathematical structures, such as topological space, lattice, Stone algebra and Nelson algebra (Chuchro 1993; Iwinski 1987; Lin and Liu 1993; Monteiro 1967; Pagliani 1993; Pomykala and Pomykala 1988; Vakarelov 1977).

3.2 The allocation view

Let $(\mathcal{A}, \vee, \wedge, \neg, 0, 1)$ and $(\mathcal{B}, \vee, \wedge, \neg, 0, 1)$ be two Boolean algebras. Under the allocation view of interval structure, one can assign elements of $\mathcal{B}$ to elements of $\mathcal{A}$ through a basic assignment j satisfying axioms (A1)-(A3), and then construct the individual intervals using equations (6) and (7). One can also directly define the intervals for every element of $\mathcal{A}$ provided that they obey the axioms of an interval structure. This latter approach was in fact adopted by Bundy (1985, 1986) in introducing incidence calculus for probabilistic reasoning.

In incidence calculus, a proposition is *not* assigned a numeric degree of belief. Instead, a proposition is associated with a list of labels called *incidences* which specify a set of situations in which the proposition is true. However, in many cases, due to the lack of knowledge, it is not always possible to assign precisely the incidences to every proposition. To resolve this problem, Bundy (1985) suggested that one may assign *lower* and *upper* bounds of the incidences to some individual propositions. A lower bound specifies those situations in which the proposition is definitely true; an upper bound specifies those situations in which the proposition could be true. There are two issues involved with such an assignment. First, one should test the consistency of the bounds. Secondly, if the given bounds are indeed consistent, one still has to infer the lower and upper bounds of the remaining propositions. Bundy (1985, 1986) proposed a set of inference axioms to perform these tasks. We will demonstrate that the *tightest* bounds can in fact be inferred. More importantly, we show that the tightest lower and upper bounds of the individual propositions form an interval structure. Since an interval structure can be equivalently defined by a basic assignment, an alternate method to compute the tightest bounds is proposed.

3.2.1 Incidence calculus

Let $\mathcal{P}$ be a set of propositions, which is closed under $\vee$, $\wedge$, and $\neg$, and let W be a set of *situations* or *possible worlds*. With regard to a situation $w \in W$, a proposition $A \in \mathcal{P}$ is either *true* or *false*. Given a proposition $A \in \mathcal{P}$, one can therefore define a subset $i(A) \subseteq W$ to indicate that A is true for all $w \in i(A)$, and A is false for all $w \notin i(A)$; $i(A)$ is referred to as the *incidence* of A. Under this interpretation, an incidence mapping $i: \mathcal{P} \longrightarrow 2^W$ should satisfy the following axioms:

$$\begin{array}{ll} \text{(IC1)} & i(\neg A) = W - i(A), \\ \text{(IC2)} & i(A \wedge B) = i(A) \cap i(B). \end{array}$$

Axiom (IC1) says that for any situation $w \in W$, if A is true, then $\neg A$ is false. Axiom (IC2) says that for any situation $w \in W$, if both A and B are true, then $A \wedge B$ is true and vice versa. In this paper, a mapping $i: \mathcal{P} \longrightarrow 2^W$ satisfying axioms (IC1) and (IC2) is referred to as an *incidence structure* which satisfies the following properties:

$$\begin{array}{ll} \text{(IC3)} & i(true) = W, \\ \text{(IC4)} & i(false) = \emptyset, \\ \text{(IC5)} & i(A \vee B) = i(A) \cup i(B). \end{array}$$

An incidence structure can be equivalently defined by another set of axioms consisting of (IC1) and (IC5). An incidence mapping i is both a $\vee$-homomorphism and a $\wedge$-homomorphism from the Boolean algebra $(\mathcal{P}, \vee, \wedge, \neg, false, true)$ to the Boolean algebra $(2^W, \cup, \cap, \neg, \emptyset, W)$. Thus, the pair (i, i) is a special kind of interval structure.

3.2.2 An interval structure formed by the tightest incidence bounds

In practice, it may be difficult to specify precisely the incidence of a proposition. Instead, one may be able to provide the *lower* and *upper bounds* of incidences for the individual propositions. In other words, one can use two mappings $\inf: \mathcal{P} \longrightarrow 2^W$ and $\sup: \mathcal{P} \longrightarrow 2^W$ to define the intervals within which the true incidences lie. A pair of lower and upper mappings $(\inf, \sup)$ is said to be *consistent* if there exists an incidence structure i such that for all $A \in \mathcal{P}$,

$$\inf(A) \subseteq i(A) \subseteq \sup(A). \tag{16}$$

In this case, we say that i is bounded by the pair $(\inf, \sup)$. A pair of mappings $\inf_0 : 2^\Theta \longrightarrow 2^W$ and $\sup_0 : 2^\Theta \longrightarrow 2^W$ is said to be the *tightest mappings* of $(\inf, \sup)$ if:

(a). $(\inf_0, \sup_0)$ is bounded by $(\inf, \sup)$, i.e., for all $A \in \mathcal{P}$,

$$\inf(A) \subseteq \inf{}_0(A) \subseteq \sup{}_0(A) \subseteq \sup(A);$$

(b). Any incidence structure i bounded by $(\inf, \sup)$ is bounded by $(\inf_0, \sup_0)$, namely,

$$(\forall A \in \mathcal{P})(\inf(A) \subseteq i(A) \subseteq \sup(A)) \Longrightarrow$$
$$(\forall A \in \mathcal{P})(\inf{}_0(A) \subseteq i(A) \subseteq \sup{}_0(A));$$

(c). No other mappings bounded by $(\inf, \sup)$ would satisfy conditions (a) and (b).

If a pair of lower and upper mappings are consistent, all the tightest bounds are unique.

Two different methods can be used to construct the tightest bounds of the individual propositions. First, a modified set of Bundy's original inference axioms can be used for testing consistency and at the same time for inferring the tightest mappings. Initially, let $\mathrm{INF}(A) = \inf(A)$ and $\mathrm{SUP}(A) = \sup(A)$ for all $A \in \mathcal{P}$. The values of INF and SUP are subsequently modified by using the following set of inference axioms:

(I1) $\mathrm{INF}(\neg A) = X \Longrightarrow \mathrm{SUP}(A) \longleftarrow \mathrm{SUP}(A) \cap (W - X);$

(I2) $\mathrm{SUP}(\neg A) = X \Longrightarrow \mathrm{INF}(A) \longleftarrow \mathrm{INF}(A) \cup (W - X);$

(I3) $\mathrm{INF}(A) = X$ and $\mathrm{INF}(B) = Y \Longrightarrow \mathrm{INF}(A \wedge B) \longleftarrow \mathrm{INF}(A \wedge B) \cup (X \cap Y);$

(I4) $\mathrm{INF}(A \wedge B) = X \Longrightarrow \mathrm{INF}(A) \longleftarrow \mathrm{INF}(A) \cup X.$

For simplicity, here we consider a version of propositional logic containing the primitive connectives $\neg$ (negation) and $\wedge$ (and). It is understood that a proposition expressed by the non-primitive connectives such as $\vee$ (or), $\Longrightarrow$ (implication), $\Longleftrightarrow$ (equivalence), and the logical constants: *true* and *false*, can be translated into a normal form containing only $\neg$ and $\wedge$. The symbol $\longleftarrow$ is an assignment operator which assigns a new value to a lower or an upper bound based on its old value. The above inference axioms are applied repeatedly until the values of the mappings INF and SUP are unchanged. Obviously, these rules will tend to enlarge the lower bounds and shrink the upper bounds. The correctness of inference rules has been discussed by Bundy (1985, 1986), which is stated in the following lemma.

Lemma 5 *Let* INF *denote the greatest lower mapping and* SUP *the smallest upper mapping inferred by axioms (I1)-(I4). An incidence structure* i *bounded by* $(\inf, \sup)$ *is also bounded by* $(\mathrm{INF}, \mathrm{SUP})$.

If the initial pair of lower and upper mappings is consistent, properties (IC3) and (IC4) imply that $\mathrm{SUP}(true) = W$ and $\mathrm{INF}(false) = \emptyset$. By rule (I3),

$$\mathrm{INF}(A) \cap \mathrm{INF}(B) \subseteq \mathrm{INF}(A \wedge B).$$

On the other hand, by rule (I4),

$$\mathrm{INF}(A \wedge B) \subseteq \mathrm{INF}(A) \cap \mathrm{INF}(B).$$

Thus,

$$\mathrm{INF}(A \wedge B) = \mathrm{INF}(A) \cap \mathrm{INF}(B).$$

That is, INF is a $\wedge$-homomorphism from $\mathcal{P}$ to 2^W. Moreover, rules (I1) and (I2) imply that $\mathrm{SUP}(A) = W - \mathrm{INF}(\neg A)$ for all $A \in \mathcal{P}$. Thus, the pair $(\mathrm{INF}, \mathrm{SUP})$ is an interval structure, i.e., the mapping $\mathrm{INF} : \mathcal{P} \longrightarrow 2^W$ is a $\wedge$-homomorphism from the Boolean algebra $(\mathcal{P}, \wedge, \vee, \neg, false, true)$ to the Boolean algebra $(2^W, \cup, \cap, \neg, \emptyset, W)$, and the mapping defined by $\mathrm{SUP}(A) = W - \mathrm{INF}(\neg A)$ is a $\vee$-homomorphism.

Theorem 2 *Let* $(\inf, \sup)$ *be a pair of consistent lower and upper mappings. The pair of the largest lower and the smallest upper mappings* $(\mathrm{INF}, \mathrm{SUP})$ *inferred from axioms (I1)-(I4) is an interval structure.*

Theorem 3 *Let* $(\inf, \sup)$ *be a pair of consistent lower and upper mappings. The pair of the largest lower and the smallest upper mappings* $(\mathrm{INF}, \mathrm{SUP})$ *inferred from axioms (I1)-(I4) is the tightest mappings of* $(\inf, \sup)$.

Recall that an interval structure can be equivalently defined by a basic assignment. The results of Theorems 2 and 3 enable us to devise an algorithm, called *Focalfinder*, to construct the tightest bounds by directly computing the basic assignment j. In Step 1, if $\inf(\neg A)$ is not assigned a value in the input, we may assume $\inf(\neg A) = \emptyset$. To ensure that every $w \in W$ will be assigned to a proposition in Step 2, we assume that $\inf(true) = W$ is in the input. It is also understood that all the trivial bounds with $\inf(A) = \emptyset$ and $\sup(A) = W$ have been eliminated from the initial input. If the algorithm prints "*inconsistent*", it indicates that the input set of bounds is not consistent. Therefore, we can also use this algorithm to test consistency.

Algorithm 1 *Focalfinder*

Input: *Let* S *denote a subset of propositions in* $\mathcal{P}$. *Suppose a lower bound* $\inf(A)$ *and an upper bound* $\sup(A)$ *of every* A *in* S *are given.*

1. For each $A \in S$ *do*

$\inf(\neg A) \leftarrow \inf(\neg A) \cup (W - \sup(A))$;

2. For each $w_k \in W$ *do*

find all the A's such that $w_k \in \inf(A)$, *say,* $A_1, A_2, \ldots, A_l$;

if $A_1 \wedge A_2 \wedge \ldots \wedge A_l = false$ *then*

print "inconsistent"; Exit;

else

$j(A_1 \wedge A_2 \wedge \ldots \wedge A_l) \leftarrow j(A_1 \wedge A_2 \wedge \ldots \wedge A_l) \cup \{w_k\}$;

(if $j(A_1 \wedge A_2 \wedge \ldots \wedge A_l)$ *is not defined, assume it to be* $\emptyset$*.)*

Output: the basic assignment j.

The following theorems show the correctness of the algorithm for computing the basic assignment.

Theorem 4 *If the pair of lower and upper mappings* $(\underline{f}, \overline{f})$ *is consistent, algorithm Focalfinder outputs the basic assignment of the tightest mappings* (INF, SUP).

Theorem 5 *Focalfinder prints "inconsistent", if and only if the input bounds are inconsistent.*

The number of computations required by *Focalfinder* depends very much on the cost of testing whether $A_1 \wedge A_2 \wedge \ldots \wedge A_l = false$. We may assume that each proposition is expressed as a disjunction of minterms, i.e., $t_1 \vee t_2 \vee \ldots \vee t_s$, where each t_j is a conjunction of atomic propositions or their negations. The complexity of this algorithm is $O(h \times |W| \times n^2 \times m^2 + h \times |W|^2)$, where h is the number of input (non-trivial) bounds, n is the number of atomic propositions, m is the maximum number of t's in the input propositions, and $|W|$ is the cardinality of W.

3.3 An example

In many practical situations, one is interested in the free Boolean algebra generated by a set of finite propositions, $\Psi = \{P_1, P_2, \ldots, P_n\}$. A minterm t of the Boolean algebra $\mathcal{P}$ generated by Ψ has the form $t = \hat{P}_1 \wedge \hat{P}_2 \wedge \ldots \wedge \hat{P}_n$, where $\hat{P}_i$ is either P_i or $\neg P_i$. Any proposition $A \in \mathcal{P}$, except the proposition $false$, can be expressed as a disjunctive normal form $t_1 \vee t_2 \vee \ldots t_k$, where t_j is a minterm such that $t_j \Rightarrow A$. The following example illustrates the procedure for constructing the basic assignment and its corresponding interval structure.

Example 2 *Suppose we have a set of two propositions* $\{P_1, P_2\}$. *Then the set of minterms generated by* $\{P_1, P_2\}$ *is:*

$$\{t_1 = P_1 \wedge P_2, \quad t_2 = P_1 \wedge \neg P_2, \quad t_3 = \neg P_1 \wedge P_2, \quad t_4 = \neg P_1 \wedge \neg P_2\}.$$

The above minterms define the following set of propositions:

$$\mathcal{P} = \left\{ \begin{array}{l} false, \\ t_1,\ t_2,\ t_3,\ t_4, \\ t_1 \vee t_2,\ t_1 \vee t_3,\ t_1 \vee t_4,\ t_2 \vee t_3,\ t_2 \vee t_4,\ t_3 \vee t_4, \\ t_1 \vee t_2 \vee t_3,\ t_1 \vee t_2 \vee t_4,\ t_1 \vee t_3 \vee t_4,\ t_2 \vee t_3 \vee t_4, \\ true = t_1 \vee t_2 \vee t_3 \vee t_4 \end{array} \right\}.$$

Let $W = \{w_1, w_2, w_3, w_4, w_5\}$*. Suppose the initial lower and upper bounds are:*

$$\begin{array}{l} \inf(t_1 \vee t_2) = \{w_1, w_4\}, \\ \inf(t_1 \vee t_3) = \{w_1, w_2\}, \\ \inf(true) = W, \\ \sup(t_3 \vee t_4) = \{w_3, w_5\}, \\ \sup(t_1 \vee t_4) = \{w_1, w_2, w_3\}. \end{array}$$

In Step 1, the two upper bounds yield:

$$\begin{array}{rcl} \inf(\neg(t_3 \vee t_4)) & = & \inf(t_1 \vee t_2) \\ & = & \inf(t_1 \vee t_2) \cup (W - \sup(t_3 \vee t_4)) \\ & = & \{w_1, w_4\} \cup (W - \{w_3, w_5\}) \\ & = & \{w_1, w_2, w_4\}, \end{array}$$

$$\begin{array}{rcl} \inf(\neg(t_1 \vee t_4)) & = & \inf(t_2 \vee t_3) \\ & = & \inf(t_2 \vee t_3) \cup (W - \sup(t_1 \vee t_4)) \\ & = & \emptyset \cup (W - \{w_1, w_2, w_3\}) \\ & = & \{w_4, w_5\}. \end{array}$$

Together with the given lower bounds, we obtain:

$$\begin{array}{l} \inf(t_1 \vee t_2) = \{w_1, w_2, w_4\}, \\ \inf(t_1 \vee t_3) = \{w_1, w_2\}, \\ \inf(t_2 \vee t_3) = \{w_4, w_5\}, \\ \inf(true) = W. \end{array}$$

In Step 2, since

$$w_1 \in \inf(t_1 \vee t_2), \qquad w_1 \in \inf(t_1 \vee t_3),$$

it follows:

$$w_1 \in j((t_1 \vee t_2) \wedge (t_1 \vee t_3)) = j(t_1).$$

Similarly, we obtain:

$$w_2 \in j(t_1), \quad w_3 \in j(true), \quad w_4 \in j(t_2), \quad w_5 \in j(t_2 \vee t_3).$$

Therefore, the resulting basic assignment j is:

$j(t_1) = \{w_1, w_2\}, \quad j(t_2) = \{w_4\}, \quad j(t_2 \vee t_3) = \{w_5\}, \quad j(true) = \{w_3\}.$

By using the formulas:

$$\mathrm{INF}(A) = \bigcup_{B \Rightarrow A} j_F(B)$$

and

$$\mathrm{SUP}(A) \ = \ \bigcup_{A \wedge B \neq \mathrm{false}} j_F(B),$$

one can construct the following tightest lower and upper bounds for every $A \in \mathcal{P}$:

$$\begin{aligned}
&\mathrm{INF}(\mathrm{false}) = \emptyset,\\
&\mathrm{INF}(t_1) = \{w_1, w_2\},\\
&\mathrm{INF}(t_2) = \{w_4\},\\
&\mathrm{INF}(t_3) = \emptyset,\\
&\mathrm{INF}(t_4) = \emptyset,\\
&\mathrm{INF}(t_1 \vee t_2) = \{w_1, w_2, w_4\},\\
&\mathrm{INF}(t_1 \vee t_3) = \{w_1, w_2\},\\
&\mathrm{INF}(t_1 \vee t_4) = \{w_1, w_2\},\\
&\mathrm{INF}(t_2 \vee t_3) = \{w_4, w_5\},\\
&\mathrm{INF}(t_2 \vee t_4) = \{w_4\},\\
&\mathrm{INF}(t_3 \vee t_4) = \emptyset,\\
&\mathrm{INF}(t_1 \vee t_2 \vee t_3) = \{w_1, w_2, w_4, w_5\},\\
&\mathrm{INF}(t_1 \vee t_2 \vee t_4) = \{w_1, w_2, w_4\},\\
&\mathrm{INF}(t_1 \vee t_3 \vee t_4) = \{w_1, w_2\},\\
&\mathrm{INF}(t_2 \vee t_3 \vee t_4) = \{w_4, w_5\},\\
&\mathrm{INF}(\mathrm{true}) = W,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{SUP}(\mathrm{false}) = \emptyset,\\
&\mathrm{SUP}(t_1) = \{w_1, w_2, w_3\},\\
&\mathrm{SUP}(t_2) = \{w_3, w_4, w_5\},\\
&\mathrm{SUP}(t_3) = \{w_3, w_5\},\\
&\mathrm{SUP}(t_4) = \{w_3\},\\
&\mathrm{SUP}(t_1 \vee t_2) = W,\\
&\mathrm{SUP}(t_1 \vee t_3\}) = \{w_1, w_2, w_3, w_5\},\\
&\mathrm{SUP}(t_1 \vee t_4) = \{w_1, w_2, w_3\},
\end{aligned}$$

$$\begin{aligned}
&\text{SUP}(t_2 \vee t_3) = \{w_3, w_4, w_5\},\\
&\text{SUP}(t_2 \vee t_4) = \{w_3, w_4, w_5\},\\
&\text{SUP}(t_3 \vee t_4) = \{w_3, w_5\},\\
&\text{SUP}(t_1 \vee t_2 \vee t_3) = W,\\
&\text{SUP}(t_1 \vee t_2 \vee t_4) = W,\\
&\text{SUP}(t_1 \vee t_3 \vee t_4\}) = \{w_1, w_2, w_3, w_5\},\\
&\text{SUP}(t_2 \vee t_3 \vee t_4) = \{w_3, w_4, w_5\},\\
&\text{SUP}(\text{true}) = W.
\end{aligned}$$

The procedure presented here can be easily extended for the construction of any interval structure over two Boolean algebras.

4 Interval Structures and Belief Functions

As a typical interval-based numeric method, belief functions have generated considerable interest in uncertainty management (Shafer 1976; Smets 1988). Belief functions are particularly useful in those situations where the input required by the Bayesian theory is not available. This section examines the relationship between belief functions and interval structures.

Given a non-empty finite set Θ, referred to as a frame of discernment, a belief function Bel is a mapping $Bel: \ 2^{\Theta} \longrightarrow [0,1]$ satisfying the following axioms:

(B1) $Bel(\emptyset) = 0$,

(B2) $Bel(\Theta) = 1$,

(B3) For every positive integer n and every collection $A_1, A_2, \ldots, A_n \in 2^{\Theta}$,

$$\sum_i Bel(A_i) - \sum_{i<j} Bel(A_i \cap A_j) \pm \ldots + (-1)^{n+1} Bel(A_1 \cap A_2 \ldots \cap A_n)$$
$$= \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} Bel(\bigcap_{i \in I} A_i) \leq Bel(A_1 \cup A_2 \ldots \cup A_n),$$

where $|\cdot|$ denotes the cardinality of a set. The corresponding *plausibility* function is defined as:

$$Pl(A) = 1 - Bel(\neg A). \tag{17}$$

We can interpret $Bel(A)$ as the belief one actually commits to A, and $Pl(A)$ as the maximum possible belief one may commit to A. It can be easily verified that $Bel(A) \leq Pl(A)$. The interval $[Bel(A), Pl(A)]$ defines the numeric uncertainty of proposition A. In the special case where $\leq$ is replaced by $=$

in axiom (B3), both the belief and plausibility functions reduce to the same probability function.

A belief function can be equivalently defined by a *basic probability assignment* $m\colon 2^{\Theta} \longrightarrow [0,1]$ satisfying the conditions:

$$
\begin{array}{ll}
\text{(M1)} & m(\emptyset) = 0, \\
\text{(M2)} & \displaystyle\sum_{A \in 2^{\Theta}} m(A) = 1.
\end{array}
$$

The belief in a proposition $A \in 2^{\Theta}$ can be expressed as:

$$
\text{(M3)} \qquad Bel(A) = \sum_{B \subseteq A} m(B).
$$

A subset $A \in 2^{\Theta}$ with $m(A) > 0$ is called a focal element. By the Möbius inversion, one can construct a basic probability assignment from a belief function using the formula:

$$
m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} Bel(B). \tag{18}
$$

Thus, a belief function can be defined by axioms (B1)-(B3) or axioms (M1)-(M3).

A pair of belief and plausibility functions may be viewed as the envelopes of a set of probability functions. A probability function P is said to be bounded by a pair of belief and plausibility functions, Bel and Pl, if $Bel(A) \leq P(A) \leq Pl(A)$ for every $A \in 2^{\Theta}$. Let Φ denote the set of all probability functions bounded by a pair of belief and plausibility functions (Bel, Pl). Dempster (1967) showed that:

$$
\begin{aligned}
Bel(A) &= \inf_{P \in \Phi} P(A), \\
Pl(A) &= \sup_{P \in \Phi} P(A).
\end{aligned} \tag{19}
$$

That is, Bel and Pl are the lower and the upper envelopes of Φ.

Bundy (1992), and Correa da Silva and Bundy (1990) studied the correspondence between incidence structures and probability functions, and the relationship between lower and upper bounds of incidence and belief and plausibility functions. Given an interval structure $(\underline{f}, \overline{f})$, if $\underline{f}$ and $\overline{f}$ satisfy both axioms (L2) and (U1), they reduce to the same incidence structure. Let I denote the set of all incidence structures bounded by an interval structure $(\underline{f}, \overline{f})$. The non-numeric lower envelope $\inf_{i \in I} i(A)$ of I is defined as the subset of W such that $\inf_{i \in I} i(A) \subseteq i(A)$ for all $i \in I$, and that for any $X \subseteq W$, if $X \subseteq i(A)$ for all $i \in I$, then $X \subseteq \inf_{i \in I} i(A)$. Similarly, the non-numeric upper envelope $\sup_{i \in I} i(A)$ of I is defined as the subset of W such that $i(A) \subseteq \sup_{i \in I} i(A)$ for all

$i \in I$, and that for any $X \subseteq W$, if $i(A) \subseteq X$ for all $i \in I$, then $\sup_{i \in I} i(A) \subseteq X$. Using these definitions, similar to equation (19), an interval structure can be expressed as (Wong, Wang and Yao 1992b):

$$\begin{aligned} \underline{f}(A) &= \inf_{i \in I} i(A), \\ \overline{f}(A) &= \sup_{i \in I} i(A). \end{aligned} \tag{20}$$

That is, the non-numeric belief $\underline{f}$ and plausibility $\overline{f}$ are the lower and the upper envelopes of I.

The next two theorems summarize the relationship between a pair of belief and plausibility functions and an interval structure.

Theorem 6 *Let W and Θ be two finite sets. Let $(\underline{f}, \overline{f})$ be an interval structure with $\underline{f}: 2^\Theta \longrightarrow 2^W$ and $\overline{f}: 2^\Theta \longrightarrow 2^W$. Suppose P is a probability function on W. Then $P(\underline{f}(A))$ is a belief function and $P(\overline{f}(A))$ is the corresponding plausibility function.*

The correspondence between the basic assignment of an interval structure and the basic probability assignment of a belief function is stated in the following corollary.

Corollary 2 *Let j be the basic assignment of an interval structure $(\underline{f}, \overline{f})$, and m the basic probability assignment of the belief function $Bel(A) = P(\underline{f}(A))$. Then*

$$m(A) = P(j(A)). \tag{21}$$

Theorem 7 *Two mappings Bel and Pl from 2^Θ to $[0, 1]$ are belief and plausibility functions, if and only if there exists an interval structure $(\underline{f}, \overline{f})$ on a finite set W, and a probability P on 2^W such that:*

$$Bel(A) = P(\underline{f}(A)), \qquad Pl(A) = P(\overline{f}(A)). \tag{22}$$

It is clear from the above analysis that belief and plausibility functions can be better understood in terms of interval structures.

5 Conclusion

In this paper, we have introduced a unified framework for representing uncertainty based on the notion of interval structures. An interval structure is defined as a pair of mappings between two Boolean algebras, which can be equivalently defined by a basic assignment. Such a structure can be considered as the non-numeric counterpart of a pair of belief and plausibility functions, while the basic assignment as the non-numeric counterpart of the basic probability assignment.

We have discussed two views for interpreting an interval structure. The compatibility view is based on a relation between the elements of two Boolean algebras. The pair of lower and upper approximations in the rough-set model is an example of an interval structure in the compatibility view. On the other hand, the set of tightest lower and upper bounds in incidence calculus provide an example of an interval structure in the allocation view. This unified approach enables us to develop a more efficient algorithm to compute the tightest incidence bounds.

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Appendix: Proofs of Theorems

In this appendix, we only give the proofs of those lemmas and theorems that do not follow trivially from the discussion in the text.

Lemma 4.**proof** For any two elements a and b of a Boolean algebra, $a \preceq b \Longleftrightarrow a \wedge b = a$. Therefore, we prove this lemma by showing that $\underline{f}(a) \wedge \overline{f}(a) = \underline{f}(a)$ as follows:

$$\begin{aligned}
\underline{f}(a) &= \underline{f}(a) \wedge 1 \\
&= \underline{f}(a) \wedge [\underline{f}(\neg a) \vee \neg \underline{f}(\neg a)] \\
&= \underline{f}(a) \wedge [\underline{f}(\neg a) \vee \overline{f}(a)] \\
&= [\underline{f}(a) \wedge \underline{f}(\neg a)] \vee [\underline{f}(a) \wedge \overline{f}(a)] \\
&= \underline{f}(a \wedge \neg a) \vee [\underline{f}(a) \wedge \overline{f}(a)] \\
&= 0 \vee [\underline{f}(a) \wedge \overline{f}(a)] \\
&= \underline{f}(a) \wedge \overline{f}(a).
\end{aligned}$$

Theorem 1.**proof** (*if*) From a given basic assignment j, we construct a mapping $\underline{f}\colon \mathcal{A} \longrightarrow \mathcal{B}$: for $a \in \mathcal{A}$,

$$\underline{f}(a) = \bigvee_{b \preceq a} j(b).$$

By the distributive properties of $\vee$ and $\wedge$,

$$\begin{aligned}
\underline{f}(a) \wedge \underline{f}(b) &= [\bigvee_{c \preceq a} j(c)] \wedge [\bigvee_{d \preceq b} j(d)] \\
&= \bigvee_{c \preceq a} \bigvee_{d \preceq b} [j(c) \wedge j(d)].
\end{aligned}$$

From (A3),

$$\bigvee_{c \preceq a} \bigvee_{d \preceq b} [j(c) \wedge j(d)] = \bigvee_{c \preceq a \wedge b} j(c) = \underline{f}(a \wedge b).$$

That is, $\underline{f}(a) \wedge \underline{f}(b) = \underline{f}(a \wedge b)$. Also, axioms (A1) and (A2) trivially imply that

$$\underline{f}(0) = 0, \quad \underline{f}(1) = 1.$$

Hence, $(\underline{f}, \overline{f})$ is an interval structure, where $\overline{f}(a) = \neg \underline{f}(\neg a)$ for all $a \in \mathcal{A}$.
(*only if*) Given an interval structure $(\underline{f}, \overline{f})$, we can construct a mapping $j\colon \mathcal{A} \longrightarrow \mathcal{B}$:

$$j(a) = \underline{f}(a) \wedge \neg [\bigvee_{b \prec a} \underline{f}(b)].$$

We will first show that $\underline{f}(a) = \bigvee_{b \preceq a} j(b)$. By the distributive properties of $\vee$ and $\wedge$, it follows:

$$\begin{aligned} j(a) \vee \{\bigvee_{b \prec a} \underline{f}(b)\} &= \{\underline{f}(a) \wedge \neg[\bigvee_{b' \prec a} \underline{f}(b')]\} \vee \{\bigvee_{b \prec a} \underline{f}(b)\} \\ &= \{\underline{f}(a) \vee [\bigvee_{b \prec a} \underline{f}(b)]\} \wedge \{\neg[\bigvee_{b' \prec a} \underline{f}(b')] \vee [\bigvee_{b \prec a} \underline{f}(b)]\} \\ &= \{\underline{f}(a) \vee [\bigvee_{b \prec a} \underline{f}(b)]\} \wedge 1 \\ &= \underline{f}(a) \vee [\bigvee_{b \prec a} \underline{f}(b)]. \end{aligned}$$

By definition, $\underline{f}\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a $\wedge$-homomorphism. From Lemma 1, we have:

$$b \preceq a \Longrightarrow \underline{f}(b) \preceq \underline{f}(a) \quad \text{and} \quad \bigvee_{b \prec a} \underline{f}(b) \preceq \underline{f}(a).$$

Hence,

$$j(a) \vee \{\bigvee_{b \prec a} \underline{f}(b)\} = \underline{f}(a) \vee \{\bigvee_{b \prec a} \underline{f}(b)\} = \underline{f}(a).$$

By applying the above equation recursively, we obtain:

$$\underline{f}(a) = \bigvee_{b \preceq a} j(b).$$

Now we show that j satisfies (A1)-(A3). Axioms (L3) and (L4) trivially imply (A1) and (A2). Since $\preceq$ is antisymmetric, we can divide the proof of (A3) into two cases: (i) $a \not\preceq b$ and $b \not\preceq a$, and (ii) $a \prec b$ (or $b \prec a$).

(i) Suppose $a \not\preceq b$ and $b \not\preceq a$. First, we want to demonstrate that $a \wedge b \prec b$ and $a \wedge b \prec a$. Since $a \wedge b \preceq b$ and $a \wedge b \preceq a$ for any $a, b \in \mathcal{A}$, it is only necessary to show that $a \wedge b \neq b$ and $a \wedge b \neq a$. Suppose $a \wedge b = a$. Since $a \wedge b \preceq b$, we have $a \preceq b$. This is a contradiction. Thus, $a \wedge b \neq a$. Similarly, we can show $a \wedge b \neq b$.

By the definition of j,

$$\begin{aligned} j(a) &= \underline{f}(a) \wedge \neg[\bigvee_{c \prec a} \underline{f}(c)] \\ &= \underline{f}(a) \wedge \neg\{[\bigvee_{\substack{c \prec a \\ c \neq a \wedge b}} \underline{f}(c)] \vee \underline{f}(a \wedge b)\} \\ &= \underline{f}(a) \wedge \{\neg[\bigvee_{\substack{c \prec a \\ c \neq a \wedge b}} \underline{f}(c)] \wedge \neg\underline{f}(a \wedge b)\} \\ &\preceq \underline{f}(a) \wedge \neg[\underline{f}(a \wedge b)]. \end{aligned}$$

Likewise,
$$j(b) \preceq \underline{f}(b) \wedge \neg \underline{f}(a \wedge b).$$
Therefore, $j(a) \wedge j(b) \preceq [\underline{f}(a) \wedge \neg \underline{f}(a \wedge b)] \wedge [\underline{f}(b) \wedge \neg \underline{f}(a \wedge b)]$. On the other hand, since $\underline{f}$ is a $\wedge$-homomorphism, for all $a, b \in \mathcal{A}$, we obtain:
$$\begin{aligned} [\underline{f}(a) \wedge \neg \underline{f}(a \wedge b)] \wedge [\underline{f}(b) \wedge \neg \underline{f}(a \wedge b)] &= [\underline{f}(a) \wedge \underline{f}(b)] \wedge \neg \underline{f}(a \wedge b) \\ &= \underline{f}(a \wedge b) \wedge \neg \underline{f}(a \wedge b) = 0. \end{aligned}$$
It follows that $j(a) \wedge j(b) = 0$.

(ii) Suppose $a \prec b$ (the case for $b \prec a$ can be proved in the same manner). In this case, we have:
$$\begin{aligned} \underline{f}(a) \wedge j(b) &= \underline{f}(a) \wedge \{\underline{f}(b) \wedge \neg[\bigvee_{c \prec b} \underline{f}(c)]\} \\ &= \underline{f}(a) \wedge \{\underline{f}(b) \wedge \neg[\bigvee_{\substack{c \prec b \\ c \neq a}} \underline{f}(c) \vee \underline{f}(a)]\} \\ &= \underline{f}(a) \wedge \underline{f}(b) \wedge [\neg \bigvee_{\substack{c \prec b \\ c \neq a}} \underline{f}(c)] \wedge \neg \underline{f}(a) = 0. \end{aligned}$$
Since $j(a) \preceq \underline{f}(a)$, $j(a) \wedge j(b) \preceq \underline{f}(a) \wedge j(b) = 0$. That is, $j(a) \wedge j(b) = 0$.

By combining the results of (i) and (ii), we can immediately conclude that (A3) holds.

Theorem 3.**proof** Showing that the pair $(\mathrm{INF}, \mathrm{SUP})$ is the tightest mappings of $(\inf, \sup)$ is equivalent to showing that given any $A \in \mathcal{P}$, there exist incidence structures i_1 and i_2 bounded by $(\mathrm{INF}, \mathrm{SUP})$, namely, for all $B \in \mathcal{P}$,
$$\begin{aligned} \mathrm{INF}(B) \subseteq i_1(B) \subseteq \mathrm{SUP}(B), \\ \mathrm{INF}(B) \subseteq i_2(B) \subseteq \mathrm{SUP}(B), \end{aligned}$$
such that
$$i_1(A) = \mathrm{INF}(A), \qquad i_2(A) = \mathrm{SUP}(A).$$

By Theorem 2, $(\mathrm{INF}, \mathrm{SUP})$ is an interval structure. Based on Theorem 1 there is a basic assignment $j\colon \mathcal{P} \longrightarrow 2^W$ satisfying (A1)-(A3). For any $C \in \mathcal{P}$ with $C \neq false$, we can express C as a disjunctive normal form, say, $C = t_1 \vee t_2 \vee \ldots \vee t_k$. Let $C' = \{t_1, t_2, \ldots, t_k\}$ denote the corresponding set of minterms. Thus, for any $B \in \mathcal{P}$, $\mathrm{INF}(B) = \bigcup_{C \Longrightarrow B} j(C) = \bigcup_{C' \subseteq B'} j(C)$. With respect to a fixed $A \in \mathcal{P}$, for each focal element $C \in \mathcal{P}$, i.e., $j(C) \neq \emptyset$, one can construct a mapping $j_C\colon C' \longrightarrow 2^{j(C)}$ satisfying the following conditions:
$$\begin{aligned} &\text{(D1)} \quad \bigcup_{t \in C'} j_C(t) = j(C), \\ &\text{(D2)} \quad j_C(t_i) \cap j_C(t_j) = \emptyset, \text{ if } i \neq j, \\ &\text{(D3)} \quad j_C(t) = \emptyset \text{ if } t \in A' \text{ and } C' \not\subseteq A'. \end{aligned}$$

For any term t, let $i_1(t) = \bigcup_{C\in\mathcal{P}} j_C(t)$ and $i_1(A) = \bigcup_{t\in A'} i_1(t)$. Clearly, i_1 is an incidence structure. Now we want to show that $\mathrm{INF}(B) \subseteq i_1(B) \subseteq \mathrm{SUP}(B)$ for all $B \in \mathcal{P}$. Suppose $w \in \mathrm{INF}(B)$, where B is any proposition in $\mathcal{P}$. There exists a $C' \subseteq B'$ such that $w \in j(C)$. By the construction of j_C, there exists a term $t \in C'$ such that $w \in j_C(t)$. Since $C' \subseteq B'$, we have $t \in B'$. By the construction of i_1, $w \in \bigcup_{C\in\mathcal{P}} j_C(t) = i_1(t)$. Thus, $w \in \bigcup_{t'\in B'} i_1(t') = i_1(B)$. That is, $\mathrm{INF}(B) \subseteq i_1(B)$ for any $B \in \mathcal{P}$. Similarly, $i_1(B) \subseteq \mathrm{SUP}(B)$.

For any $w \in i_1(A)$, there exists a term $t \in A'$ such that $w \in i_1(t) = \cup_{C\in\mathcal{P}} j_C(t)$. That is, there exists a $C \in \mathcal{P}$ such that $w \in j_C(t)$. By (D3), $C' \subseteq A'$. By construction, $w \in j(C)$. Therefore, $w \in \mathrm{INF}(C) \subseteq \mathrm{INF}(A)$, namely, $w \in \mathrm{INF}(A)$. It follows that $\mathrm{INF}(A) = i_1(A)$.

Similarly, we can show that given any $A \in \mathcal{P}$, there exists an incidence structure i_2 bounded by $(\mathrm{INF}, \mathrm{SUP})$ such that $i_2(A) = \mathrm{SUP}(A)$.

Theorem 4.**proof** Obviously, $j(\text{false}) = \emptyset$. In Step 2 of *Focalfinder*, each $w_k \in W$ is uniquely assigned to $j(A)$ for some $A \in \mathcal{P}$. Thus, $\bigcup_{A\in\mathcal{P}} j(A) = W$. Also, for $A \neq B$, $j(A) \cap j(B) = \emptyset$. That is, j satisfies (A1)-(A3). Now, given any $A \in \mathcal{P}$, we want to show that $j(B) \subseteq \mathrm{INF}(A)$ for any $B' \subseteq A'$. By Step 2, for any $w \in j(B)$, there exist $A_1, A_2, \ldots, A_l$ in $\mathcal{P}$ such that $B = A_1 \wedge A_2 \wedge \ldots \wedge A_l$ and $w \in \inf(A_i)$, $i = 1, 2, \ldots, l$. On the other hand, by applying axiom (I3), we can conclude that $w \in \mathrm{INF}(B)$. By (I3) again, $w \in \mathrm{INF}(A)$. Thus, $j(B) \subseteq \mathrm{INF}(A)$ for any $B' \subseteq A'$. Let $\underline{f}(A) = \bigcup_{B'\subseteq A'} j(B)$ and $\overline{f}(A) = W - \underline{f}(\neg A)$. This pair $(\underline{f}, \overline{f})$ forms an interval structure and $\underline{f}(A) \subseteq \mathrm{INF}(A) \subseteq \mathrm{SUP}(A) \subseteq \overline{f}(A)$. Suppose there exists a $A \in \mathcal{P}$ such that $\underline{f}(A) \subset \mathrm{INF}(A)$. From the proof of Theorem 3, for any interval structure $(\underline{f}, \overline{f})$, there exists an incidence structure i_1 bounded by the interval structure such that $\underline{f}(A) = i_1(A)$ for a given $A \in \mathcal{P}$. This means that i_1 is bounded by the interval structure $(\underline{f}, \overline{f})$, but not bounded by $(\mathrm{INF}, \mathrm{SUP})$.

Given $A \in \mathcal{P}$, it can be seen from Step 2 of *Focalfinder* that for every $w \in \inf(A)$, there exists a $B' \subseteq A'$ such that $w \in j(B)$. Thus, $w \in \bigcup_{B'\subseteq A'} j(B) = \underline{f}(A)$. That is, $\inf(A) \subseteq \underline{f}(A)$. Similarly, we can show $\overline{f}(A) \subseteq \sup(A)$. Therefore, we obtain:

$$\inf(A) \subseteq \underline{f}(A) = i_1(A) \subset \mathrm{INF}(A) \subseteq \mathrm{SUP}(A) \subseteq \overline{f}(A) \subseteq \sup(A).$$

Clearly, $i_1(A)$ is bounded by $(\inf, \sup)$ but not bounded by $(\mathrm{INF}, \mathrm{SUP})$. By Lemma 5, this is a contradiction. Thus, $\underline{f}(A) = \mathrm{INF}(A)$ for every $A \in \mathcal{P}$. Similarly, one can show that $\overline{f}(A) = \mathrm{SUP}(A)$ for every $A \in \mathcal{P}$. That is, j is the basic assignment of the pair of tightest mappings $(\mathrm{INF}, \mathrm{SUP})$.

Theorem 5.**proof** (*if*) Suppose the initial assignment is inconsistent and *Focalfinder* does not print "*inconsistent*". Then the algorithm must output a basic assignment. Based on the proof of Theorems 3, one can construct at least one incidence structure bounded by the initial assignment. This contradicts the assumption that the initial assignment is inconsistent. Therefore, if the initial assignment is inconsistent, *Focalfinder* must print "*inconsistent*".

(*only if*) Suppose that the algorithm prints "*inconsistent*". In Step 2 of *Focalfinder*, there exist a $w \in W$ and $A_1, A_2, \ldots, A_l$ in $\mathcal{P}$ such that $A_1 \wedge A_2 \wedge \ldots \wedge A_l = false$ and $w \in \inf(A_i)$, $i = 1, 2, \ldots, l$. By rule (I3), this implies that the lower bound of $false$ contains w. However, there is no incidence structure that would be bounded by such a lower bound. Thus, the initial assignment is inconsistent.

Theorem 6.**proof** Since $(\underline{f}, \overline{f})$ is an interval structure, by Theorem 1 there exists a basic assignment $j\colon 2^\Theta \longrightarrow 2^W$. Given a probability measure P on 2^W, one can define a function $m\colon 2^\Theta \longrightarrow [0,1]$ as:

$$m(A) = \sum_{w \in j(A)} P(\{w\}) = P(j(A)).$$

This function satisfies (M1) and (M2). Thus, it is a basic probability assignment. By definition,

$$Bel(A) = P(\underline{f}(A)) = \sum_{w \in \underline{f}(A)} P(\{w\}).$$

From the properties of a basic assignment and the relation $\underline{f}(A) = \bigcup_{B \subseteq A} j(B)$, we obtain:

$$\sum_{w \in \underline{f}(A)} P(\{w\}) = \sum_{B \subseteq A} \sum_{w \in j(B)} P(\{w\}) = \sum_{B \subseteq A} m(B).$$

Therefore, Bel is a belief function. Since $\overline{f}(A) = W - \underline{f}(\neg A)$, $Pl(A) = P(\overline{f}(A))$ is the corresponding plausibility function.

Theorem 7.**proof** The *if* part of the proof follows trivially from Theorem 6. The *only if* part of the proof is given below.

Suppose $Bel\colon 2^\Theta \longrightarrow [0,1]$ is a belief function. Using the focal elements of Bel, we can construct a finite set W:

$$W = \{w_A \mid m(A) \neq 0\},$$

and a probability function P on W:

$$P(\{w_A\}) = m(A).$$

Using the basic probability assignment m, a basic assignment $j\colon 2^\Theta \longrightarrow 2^W$ can be defined as:

$$j(A) = \begin{cases} \{w_A\} & \text{if } m(A) \neq 0, \\ \emptyset & \text{if } m(A) = 0. \end{cases}$$

Let $\underline{f}(A) = \bigcup_{B \subseteq A} j(B)$ and $\overline{f}(A) = W - \underline{f}(\neg A)$. By Theorem 1, $(\underline{f}, \overline{f})$ is an interval structure. Moreover,

$$P(\underline{f}(A)) = \sum_{B \subseteq A} P(j(B))$$

$$
\begin{aligned}
&= \sum_{B \subseteq A} P(\{w_B\}) \\
&= \sum_{B \subseteq A} m(B) \quad = \quad Bel(A)
\end{aligned}
$$

and

$$
\begin{aligned}
P(\overline{f}(A)) &= P(W - \underline{f}(\neg A)) \\
&= 1 - P(\underline{f}(\neg A)) \\
&= 1 - Bel(\neg A) = Pl(A).
\end{aligned}
$$