Fundamenta Informaticae XXI (2010) 1001–1018
IOS Press

Set-theoretic Approaches to Granular Computing

Yiyu Yao
Department of Computer Science
University of Regina, Regina, Saskatchewan, Canada S4S 0A2
yyao@cs.uregina.ca

Nan Zhang *
Department of Computer Science and Technology
Tongji University, Shanghai, China 201804
and
Department of Computer Science
University of Regina, Regina, Saskatchewan, Canada S4S 0A2
zhangnan0851@163.com

Duoqian Miao
Department of Computer Science and Technology
Tongji University, Shanghai, China 201804
miaoduoqian@163.com

Feifei Xu
Department of Computer Science and Technology
Tongji University, Shanghai, China 201804
xufeifei1983@hotmail.com

Abstract. A framework is proposed for studying a particular class of set-theoretic approaches to granular computing. A granule is a subset of a universal set, a granular structure is a family of subsets of the universal set, and relationship between granules is given by the standard set-inclusion

*Address of corresponding author: Room 501, School of Electronics and Information Engineering, Tongji University, Shanghai, China 201804.

relation. By imposing different conditions on the family of subsets, we can define several types of granular structures. A number of studies, including rough set analysis, formal concept analysis and knowledge spaces, adopt specific models of granular structures. The proposed framework therefore provides a common ground for unifying these studies. The notion of approximations is examined based on granular structures.

Keywords: Granular computing, Granular structures, Rough set analysis, Formal concept analysis, Knowledge spaces

1. Introduction

The study of granular computing focuses on a general theory and methodology for problem solving and information processing by considering multiple levels of granularity [1, 10, 13, 15, 16, 17, 21, 33, 39]. In a comprehensive bibliometrics analysis of publications of granular computing in its first ten years, JT Yao [26, 27] shows that granular computing research has been dominated by rough sets and fuzzy sets, and that there is a need to go beyond and to "broaden and deepen the study of granular computing." As a matter of fact, the triarchic theory of granular computing, proposed by the first author [31, 32, 33], indeed attempts to promote granular computing as a new field of study in its own right.

The triarchic theory of granular computing consists of three perspectives: granular computing as structured thinking, as structured problem solving, and as structured information processing [31, 32, 33]. One of its central notions is hierarchical multilevel granular structures defined by granules and levels. Each granule represents a focal point, or a unit of discussion on a particular level; each level is populated with granules of similar grain-sizes (i.e., granularity) or similar features; all levels are (partially) ordered according to their granularity. Problem solving can be approached as top-down, bottom-up or middle-out [22] processes based on a granular structure.

The formulation and interpretation of granular structures are much application dependent. Different vocabulary, terminology or language may be used for description at different levels. Nevertheless, one may study some mathematical models independent of specific applications. Keet [12, 13] proposes a logic based formal theory of granularity and gives a taxonomy of types of granularity. In this paper, we propose a framework for set-theoretic approaches to granular computing. In other words, we focus mainly on set-theoretic formulation, interpretation and representation of multilevel granular structures. Within the proposed framework, we investigate granular structures used in several studies.

This is an extended version of the paper "Set-theoretic Models of Granular Structure" [37] presented at RSKT 2010. Section 2 contains mainly the materials about set-theoretic models that have been covered in the conference paper. A basic model of a granular structure is given by a poset $(G, \subseteq)$, where G is a family of subsets of a universal set and $\subseteq$ is the set-inclusion relation. By imposing different sets of conditions on G, we derive seven sub-models of granular structures. The two new sections investigate the construction and applications of granular structures. Section 3 examines specific models of granular structures used in three separate studies, namely, rough set analysis [18, 19, 20], formal concept analysis [9, 23, 24] and knowledge spaces [6, 7, 8]. This provides an important step for integrating the three theories. By generalizing ideas of rough set analysis, Section 4 studies the notions of lower and upper approximations based on a granular structure [14, 25, 28, 29, 30, 35, 36]. The results may be generalized to approximations based on other abstract structures studied by Ciucci [4, 5].

2. Set-theoretic models of granular structures

This section introduces and investigates granules, granule structures and classes of granular structures.

2.1. Granules and granular structures

Two key notions of granular computing are granules and a hierarchical granular structure formed by a family of granules. In constructing a unified set-theoretic model of granular structures, we assume that a granule is a subset of a universal set and a granular structure is constructed based on the standard set-inclusion relation on a family of subsets of the universal set.

Definition 2.1. Let U denote a finite nonempty universal set. A subset $g \in 2^U$ is called a granule, where 2^U is the power set of U.

The power set 2^U consists of all possible granules formed from a universal set U. The standard set-inclusion relation $\subseteq$ defines a partial order on 2^U, which leads to sub-super relationship between granules.

Definition 2.2. For $g, g' \in 2^U$, if $g \subseteq g'$, we call g a sub-granule of g' and g' a super-granule of g.

Under the partial order $\subseteq$, the empty set $\emptyset$ is the smallest granule and the universe U is the largest granule. When constructing a granular structure, we may consider a family G of subsets of U and an order relation on G.

Definition 2.3. Suppose $G \subseteq 2^U$ is a nonempty family of subsets of U. The poset $(G, \subseteq)$ is called a granular structure, where $\subseteq$ is the set-inclusion relation.

By the relation $\subseteq$, we can arrange granules in G into a hierarchical multilevel granular structure. The relation $\subseteq$ is an example of partial orders. In general, one may consider any partial order on G and the corresponding poset $(G, \preceq)$. For simplicity, we consider only the poset $(G, \subseteq)$, but the argument can be easily applied to any poset.

2.2. Models of granular structures

A granular structure is formed by a family of granules. Depending on its properties, one can broadly classify set-theoretic models of granular structures into lattice-based models and set-based models.

2.2.1. An overview

The structure $(G, \subseteq)$ gives rise to the weakest set-theoretic model in which a granule is a subset of a universe, and a granular structure is a family of subsets of the universe. We denote this basic model by the pair $\mathcal{M}_0 = (U, G)$. In constructing the basic model, we only assume that $G \neq \emptyset$ and there are no other constraints. The family G does not have to be closed with respect to any set-theoretic operations. The structure of G is only a partial order defined by $\subseteq$.

The granular structure $(G, \subseteq)$ of model $\mathcal{M}_0$ is a substructure of $(2^U, \subseteq)$. Each granule in G represents a focal point of our discussion. The family G represents all focal points of our discussion. The

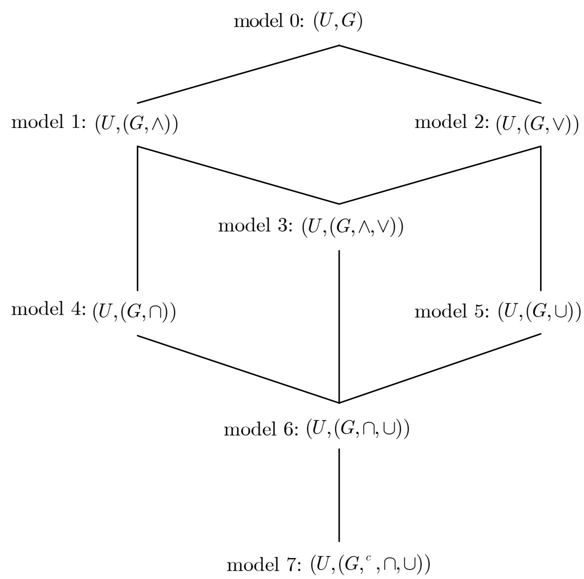

Figure 1. Models of granular structures

construction and operation of granules depend on particular applications. By imposing extra conditions on G, we can derive more specific models from the basic model. Fig.1 summarizes eight models of granular structures. A line connecting two models in Fig.1 indicates the sub-model relationship and relations that can be obtained from the transitivity are not explicitly drawn. For example, $\mathcal{M}_1$ is a sub-model of $\mathcal{M}_0$ and $\mathcal{M}_4$ is a sub-model of $\mathcal{M}_1$. It follows that $\mathcal{M}_4$ is a sub-model of $\mathcal{M}_0$.

For the convenience of discussion, we divide models in Fig.1 into two groups. One group is lattice-based models of granular structures, the other group is set-based models of granular structures. The three models $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ are lattice-based models. They correspond to meet-semilattice, join-semilattice and lattice, respectively, where the symbols $\wedge$ and $\vee$ are lattice meet and joint operations. The meet $\wedge$ and join $\vee$ may not necessarily coincide with the set-theoretic operations $\cap$ and $\cup$. When they are in fact set intersection ($\cap$) and union ($\cup$), we have the three set-theoretic models $\mathcal{M}_4$, $\mathcal{M}_5$ and $\mathcal{M}_6$. The most specific model is the one in which G is closed under all three set-theoretic operations, where ^{c} denotes set complement.

The models in Fig.1 represent hierarchical granular structures that are commonly used in many studies. A mixture of models $\mathcal{M}_2$ and $\mathcal{M}_4$ are used in formal concept analysis [9, 23, 24], where a granular structure $(U, (G, \cap, \vee))$ is used; while the meet is given by the set intersection, the join is defined differently. Model $\mathcal{M}_5$ is used in the study of knowledge spaces [6, 7, 8]. Model $\mathcal{M}_7$ is used in Pawlak rough set analysis [18, 19]. All these models are considered in the generalized rough set models [28, 29].

2.2.2. Lattice-based models of granular structures

In a granular structure $(G, \subseteq)$, where $G \subseteq 2^U$ and $\subseteq$ is the set-inclusion relation, the relation $\subseteq$ is a partial order (i.e., $\subseteq$ is reflexive, antisymmetric and transitive). One can derive three lattice-based models with respect to the partial order.

Definition 2.4. Suppose $(G, \subseteq)$ is a granular structure. For a pair of granules $a, b \in G$, a granule $l \in G$ is called a lower bound of a and b if $l \subseteq a$ and $l \subseteq b$; a granule $u \in G$ is called an upper bound of a and b if $a \subseteq u$ and $b \subseteq u$. In addition, l is called the greatest lower bound (glb) of a and b, if $k \subseteq l$ for any lower bound of k of a and b; u is called the least upper bound (lub) of a and b if $u \subseteq k$ for any upper bound k of a and b.

For an arbitrary pair of granules in G, their lower bounds, the greatest lower bound, upper bounds or the least upper bound may not exist in G. If the greatest lower bound exists, it is unique and is denoted by $a \wedge b$; if the least upper bound exists, it is unique and is denoted by $a \vee b$. Based on these notions, we immediately obtain two models of granular structures.

Definition 2.5. A granular structure $(G, \subseteq)$ is a meet-semilattice, denoted by $(G, \wedge)$, if the greatest lower bound always exists in G for any pair of granules in G. A granular structure $(G, \subseteq)$ is a join-semilattice, denoted by $(G, \vee)$, if the least upper bound always exists in G for any pair of granules in G.

In model $\mathcal{M}_1$, a granular structure $(G, \subseteq)$ is a meet-semilattice $(G, \wedge)$, $a \wedge b$ is the largest granule contained by both a and b. Since G is not necessarily closed under set intersection, $\wedge$ is not necessarily the same as $\cap$. Similarly, in model $\mathcal{M}_2$ a granular structure $(G, \subseteq)$ is a join-semilattice $(G, \vee)$, $a \vee b$ is the smallest granule in G that contains both a and b. Again, $\vee$ is not necessarily the same as $\cup$.

In a meet-semilattice granular structure $(G, \wedge)$, for a pair of granules p and g, if $p \cap g \in G$, then $p \wedge g = p \cap g$. In a join-semilattice granular structure $(G, \vee)$, for a pair of granules p and g, if $p \cup g \in G$, then $p \vee g = p \cup g$.

Example 2.1. Suppose $\mathcal{M}_1 = (U, (G_1, \wedge))$, where $U = \{a, b, c, d, e\}$, $G_1 = \{\{c\}, \{a, b, c\}, \{b, c, d\}\}$. It can be easily verified that $(G_1, \wedge)$ is a meet-semilattice. Consider two granules, $p = \{a, b, c\}$ and $q = \{b, c, d\} \in G_1$, The glb of p and q is $p \wedge q = \{c\}$. On the other hand, the intersection of p and q is $p \cap q = \{b, c\}$, which is not in G_1.

Example 2.2. Suppose $\mathcal{M}_2 = (U, (G_2, \vee))$, where $U = \{a, b, c, d, e\}$, $G_2 = \{\{a, b\}, \{c, d\}, \{a, b, c, d, e\}\}$. The granular structure $(G_2, \vee)$ is a join-semilattice. Consider two granules, $p = \{a, b\}$ and $q = \{c, d\} \in G_2$, The lub of p and q is $p \vee q = \{a, b, c, d, e\}$. On the other hand, the union of p and q is $p \cup q = \{a, b, c, d\}$, which is not in G_2.

The operators $\wedge$ and $\vee$ defined based on the partial order $\subseteq$ are referred to as the meet and join operators of semilattices. If both $\wedge$ and $\vee$ are defined for a granular structure, one can derive a lattice.

Definition 2.6. A granular structure $(G, \subseteq)$ is a lattice, denoted by $(G, \wedge, \vee)$, if both the greatest lower bound and the least upper bound always exist in G for any pair of granules in G.

In $\mathcal{M}_3$, a granular structure is a lattice, in which $\wedge$ and $\vee$ is not necessarily the same as set intersection $\cap$ and $\cup$, respectively.

Example 2.3. Suppose $\mathcal{M}_3 = (U, (G_3, \wedge, \vee))$, where $U = \{a, b, c, d, e\}$, $G_3 = \{\{c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d, e\}\}$. The granular structure $(G_3, \wedge, \vee)$ is a lattice. Consider two granules, $p = \{a, b, c\}$ and $q = \{b, c, d\} \in G_3$, the glb of p and q is $p \wedge q = \{c\}$. The intersection of p and q is $p \cap q = \{c, d\}$, which is not in G_3. The lub of p and q is $p \vee q = \{a, b, c, d, e\}$. The union of p and q is $p \cup q = \{a, b, c, d\}$, which is not in G_3.

2.2.3. Set-based models of granular structures

Set-based models of granular structures are special cases of lattice-based models, where the lattice meet $\wedge$ coincides with set intersection $\cap$ and lattice join $\vee$ coincides with set union $\cup$. In other words, G is closed under set intersection and union, respectively. We immediately obtain set-based model $\mathcal{M}_4$, $\mathcal{M}_5$ and $\mathcal{M}_6$.

Definition 2.7. A granular structure $(G, \subseteq)$ is a $\cap$-closed granular structure, denoted by $(G, \cap)$, if the intersection of any pair of granules of G is in G; a granular structure $(G, \subseteq)$ is a $\cup$-closed granular structure, denoted by $(G, \cup)$, if the union of any pair of granules of G is in G; a granular structure $(G, \subseteq)$ is a $(\cap, \cup)$-closed granular structure, denoted by $(G, \cap, \cup)$, if both the intersection and union of any pair of granules of G are in G.

Example 2.4. Suppose $\mathcal{M}_4 = (U, (G_4, \cap))$, where $U = \{a, b, c, d, e\}$, $G_4 = \{\{b, c\}, \{a, b, c\}, \{b, c, d\}\}$, $\mathcal{M}_4$ is a model of $\cap$-closed granular structure. Suppose $\mathcal{M}_5 = (U, (G_5, \cup))$, where $U = \{a, b, c, d, e\}$, $G_5 = \{\{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$, $\mathcal{M}_5$ is a model of $\cup$-closed granular structure. Suppose $\mathcal{M}_6 = (U, (G_6, \cap, \cup))$, where $U = \{a, b, c, d, e\}$, $G_6 = \{\{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}\}$, $\mathcal{M}_6$ is a model of $(\cap, \cup)$-closed granular structure.

In the granular structure $(G, \cap)$, the largest granule U may not be in G. If $U \in G$, the granular structure $(G, \cap)$ is a closure system.

Definition 2.8. A $\cap$-closed granular structure $(G, \cap)$ is a closure system, if $U \in G$.

A closure system that is closed under set union is referred to as a $\cup$-closure system [2]. Given a granular structure $(G, \cap)$ that is a closure system, its dual system $(G^c, \cup) = (\{g^c \mid g \in G\}, \cup)$ contains $\emptyset$ and is closed under set union.

Definition 2.9. A granular structure $(G, \subseteq)$ is a Boolean algebra, denoted by $(G,^c, \cap, \cup)$, if G is closed under set complement, intersection and union, respectively.

The Boolean algebra $(G,^c, \cap, \cup)$ is an σ-algebra used in Pawlak rough sets.

2.3. Characterization of classes of granular structures

Different models of granular structures have different properties. To characterize and classify these models, we introduce the following list of axioms:

$\mathcal{S}_0$: $G \neq \emptyset$
$\mathcal{S}_1$: $(a \in G, b \in G) \implies$ glb$(a, b) = a \wedge b$ exists in G
$\mathcal{S}_2$: $(a \in G, b \in G) \implies$ lub$(a, b) = a \vee b$ exists in G
$\mathcal{S}_3$: $(a \in G, b \in G) \implies a \cap b \in G$
$\mathcal{S}_4$: $(a \in G, b \in G) \implies a \cup b \in G$
$\mathcal{S}_5$: $a \in G \implies a^c \in G$
$\mathcal{S}_6$: $\emptyset \in G$
$\mathcal{S}_7$: $U \in G$

Axioms $\mathcal{S}_1$ and $\mathcal{S}_2$ define lattice-based structures; axioms $\mathcal{S}_3$ and $\mathcal{S}_4$ define set-based structures; axioms $\mathcal{S}_3$ and $\mathcal{S}_4$ are the special cases of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. These axioms are not independent. For example, $(\mathcal{S}_5, \mathcal{S}_6) \Longrightarrow \mathcal{S}_7$, $(\mathcal{S}_5, \mathcal{S}_7) \Longrightarrow \mathcal{S}_6$, $(\mathcal{S}_4, \mathcal{S}_5) \Longrightarrow \mathcal{S}_3$, and $(\mathcal{S}_3, \mathcal{S}_5) \Longrightarrow \mathcal{S}_4$.

The family of models in Fig.1 is characterized by these axioms as follows:

$\mathcal{M}_0$: $\mathcal{S}_0$
$\mathcal{M}_1$: $\mathcal{S}_0, \mathcal{S}_1$
$\mathcal{M}_2$: $\mathcal{S}_0, \mathcal{S}_2$
$\mathcal{M}_3$: $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$
$\mathcal{M}_4$: $\mathcal{S}_0, \mathcal{S}_3$
$\mathcal{M}_5$: $\mathcal{S}_0, \mathcal{S}_4$
$\mathcal{M}_6$: $\mathcal{S}_0, \mathcal{S}_3, \mathcal{S}_4$
$\mathcal{M}_7$: $\mathcal{S}_0, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$

Additional models can also be obtained. For example, a closure system is defined by $\mathcal{S}_3$ and $\mathcal{S}_7$. A $\cup$-closure system is defined by $\mathcal{S}_3$, $\mathcal{S}_4$ and $\mathcal{S}_7$, and granular structure used in formal concept analysis is defined by $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_6$. Some of the models discussed in this section have been studied in the context of rough set theory as generalized rough set models.

3. Constructions of granular structures

In the last section, we only study properties of different classes of granular structures. We now turn our attention to the construction of a few of them to show that they are, in fact, useful in several studies.

3.1. Granular structure $(U, (\sigma(U/E), \emptyset, U,^c, \cap, \cup))$ in rough set analysis

The theory of rough sets concerns the analysis of data given in a tabular form [18, 19, 34]. An information table is used to represent the relationships between a finite set of objects and a finite set of attributes. Formally, an information table is defined by:

$$M = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}),$$

where

U : is a finite nonempty set of objects,
At : is a finite nonempty set of attributes,
V_a : is a nonempty set of values for an attribute $a \in At$,
I_a : $U \longrightarrow V_a$ is an information function.

The information function I_a maps an object into a value in V_a. That is, for a pair of object $x \in U$ and attribute $a \in At$, $I_a(x) \in V_a$ is the value of x on the attribute a.

A basic notion of rough set is a granulation of the set of objects based on their descriptions. For a subset of attributes $P \subseteq At$, one can define an equivalence relation E_P on U as:

$$xE_Py \Longleftrightarrow (\forall a \in P)\ I_a(x) = I_a(y).$$

That is, two objects x and y are equivalent, if and only if they have the same values on all attributes in P. The equivalence class containing x is given by:

$$[x]_{E_P} = \{y \mid xE_P y\}.$$

The equivalence relation E_P partitions U into a family of disjoint subsets called a partition of universe and is denoted by U/E_P.

The partition U/E_P can be viewed as a family of basic granules. This can be explained as follows [19, 38]. Given an attribute-value pair (a, v), where $a \in P$ and $v \in V_a$. One can define an atomic formula $a = v$. The meaning of $a = v$ is a subset of objects defined by:

$$m(a = v) = \{x \in U \mid I_a(x) = v\}.$$

The set $m(a = v) \subseteq U$ is a granule. With respect to an object $x \in U$, we can construct a logic formula $\bigwedge_{a \in P} a = I_a(x)$. Its meaning is given by:

$$m(\bigwedge_{a \in P} a = I_a(x)) = \bigcap_{a \in P} m(a = I_a(x)) = [x]_{E_P}.$$

Thus, the equivalence class $[x]_{E_P}$ is a basic granule defined by the logic formula $\bigwedge_{a \in P} a = I_a(x)$.

By taking the union of a family of equivalence classes, we can construct a granule that is defined by the disjunction of logic formulas of these equivalence classes. The granular structure used in rough set analysis is the family of all definable granules given by:

$$\sigma(U/E) = \{X \subseteq U \mid X = \bigcup_{A \in F} A, F \subseteq 2^{U/E}\},$$

where F is a family of equivalence classes. It can be easily verified that $\sigma(U/E)$ contains $\emptyset$ and U, and is closed under set complement, intersection and union. The family $\sigma(U/E)$ is an σ-algebra. It satisfies $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_6$ and $\mathcal{S}_7$. This model of granular structures is denoted by $(U, (\sigma(U/E), \emptyset, U, ^c, \cap, \cup))$ with the minimal element $\emptyset$ and maximal element U.

Table 1. An information table

	a_1	a_2	a_3
x_1	1	1	1
x_2	1	1	1
x_3	2	1	2
x_4	0	2	0
x_5	0	2	0

Example 3.1. Table 1 is an example of an information table. Suppose $U = \{x_1, x_2, x_3, x_4, x_5\}$, $At = \{a_1, a_2, a_3\}$. For a subset of attributes, $P = \{a_1, a_2\}$, it defines an equivalence relation:

$$x_1E_Px_1,\ x_1E_Px_2,\ x_2E_Px_1,\ x_2E_Px_2,\ x_3E_Px_3,\ x_4E_Px_4,\ x_4E_Px_5,\ x_5E_Px_4,\ x_5E_Px_5.$$

The corresponding partition $U/E_P = \{\{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}\}$. From object x_2, with respect to $P = \{a_1, a_2\}$, we can construct a logic formula $a_1 = 1 \wedge a_2 = 1$ that defines the granule $m(a_1 = 1 \wedge a_2 = 1) = m(a_1 = 1) \cap m(a_2 = 1) = \{x_1, x_2\} \cap \{x_1, x_2, x_3\} = \{x_1, x_2\} = [x_1]_{E_P}$. Consider a family of equivalence classes $F = \{[x_1]_{E_P}, [x_4]_{E_P}\}$, we can construct a definable granule $[x_1]_{E_P} \cup [x_4]_{E_P} = \{x_1, x_2\} \cup \{x_4, x_5\} = \{x_1, x_2, x_4, x_5\}$, which is defined by the formula $(a_1 = 1 \wedge a_2 = 1) \vee (a_1 = 0 \wedge a_2 = 2)$. By taking the union of equivalence classes, the family of all definable granules is given by:

$$\sigma(U/E_P) = \{\emptyset, \{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4, x_5\}, \{x_3, x_4, x_5\}, U\}.$$

It is closed under set complement, intersection and union.

3.2. Granular structure $(U, (Ext(L), \cap, \vee))$ in formal concept analysis

Formal concept analysis studies concept formulation and visualization by using the notion of a formal context [9, 23, 24]. A formal context represents the relationships between a set of objects and a set of properties or attributes. It can be viewed as an information table in which the domain of every attribute is binary, namely, the presence or the absence of the corresponding property. Alternatively, a formal context can be expressed as a binary relation between a set of objects and a set of properties [30, 35].

Let U and V be two finite sets. Elements of U are called objects, and elements of V are called properties. The relationships between objects and properties are described by a binary relation R from U to V, which is a subset of the Cartesian product $U \times V$. For a pair of an object $x \in U$ and a property $y \in V$, if $(x, y) \in R$, or xRy, we say that x has the property y, or the property y is possessed by object x. The triplet (U, V, R) is called a formal context. Based on the binary relation, we associate a set of properties to an object $x \in U$:

$$xR = \{y \in V \mid xRy\} \subseteq V.$$

It is the set of properties possessed by x. Similarly, property y is possessed by the set of objects:

$$Ry = \{x \in U \mid xRy\} \subseteq U.$$

By extending these notations, we can establish relationships between subsets of objects and subsets of properties. This leads to two operators, one from 2^U to 2^V and the other from 2^V to 2^U.

Definition 3.1. For a subset of objects $X \subseteq U$, we associate it with a set of properties:

$$\begin{aligned} X^* &= \{y \in V \mid \forall x \in U (x \in X \Longrightarrow xRy)\} \\ &= \{y \in V \mid X \subseteq Ry\} \\ &= \bigcap_{x \in X} xR. \end{aligned}$$

For a subset of properties $Y \subseteq V$, we associate it with a set of objects:

$$\begin{aligned} Y^* &= \{x \in U \mid \forall y \in V (y \in Y \Longrightarrow xRy)\} \\ &= \{x \in U \mid Y \subseteq xR\} \\ &= \bigcap_{y \in Y} Ry. \end{aligned}$$

By definition, $\{x\}^* = xR$ is the set of attributes possessed by the object x, and $\{y\}^* = Ry$ is the set of objects having attribute y. For a set of objects $X \subseteq U$, X^* is the maximal set of properties shared by all objects in X. Similarly, for a set of attributes $Y \subseteq V$, Y^* is the maximal set of objects that have all attributes in Y. In formal concept analysis, one is interested in a pair of a set of objects and a set of properties that define each other.

Definition 3.2. A pair (X, Y) of a set of objects $X \subseteq U$ and a set of properties $Y \subseteq V$ is called a formal concept if

$$X = Y^* \text{ and } X^* = Y.$$

The set of objects X is called the extension of the formal concept (X, Y) and the set of properties is called the intension.

For $X, X_1, X_2 \subseteq U$ and $Y, Y_1, Y_2 \subseteq V$, it can be verified that the operators satisfy the following properties:

$$\begin{array}{ll}
(1) & X_1 \subseteq X_2 \Longrightarrow X_1^* \supseteq X_2^*, \\
 & Y_1 \subseteq Y_2 \Longrightarrow Y_1^* \supseteq Y_2^*; \\
(2) & X \subseteq X^{**}, \\
 & Y \subseteq Y^{**}; \\
(3) & X^{***} = X^*, \\
 & Y^{***} = Y^*; \\
(4) & (X_1 \cup X_2)^* = X_1^* \cap X_2^*, \\
 & (Y_1 \cup Y_2)^* = Y_1^* \cap Y_2^*.
\end{array}$$

It follows that the family of all formal concepts forms a complete lattice called a concept lattice. The meet and join of the lattice is given by [9, 23]:

$$(X_1, Y_1) \wedge (X_2, Y_2) = (X_1 \cap X_2, (Y_1 \cup Y_2)^{**}),$$
$$(X_1, Y_1) \vee (X_2, Y_2) = ((X_1 \cup X_2)^{**}, Y_1 \cap Y_2).$$

Let L denote the lattice formed by all formal concepts.

To construct a granular structure in U, we can collect the extensions of all formal concepts:

$$Ext(L) = \{X \subseteq U \mid Y \subseteq V, (X, Y) \in L\}.$$

By the lattice-theoretic operations, we can conclude that $Ext(L)$ is closed under set intersection. In addition, for two sets of objects $X_1, X_2 \in Ext(L)$, their least upper bound is given by $X_1 \vee X_2 = (X_1 \cup X_2)^{**}$. Thus, the granular structure $Ext(L)$ satisfies $\mathcal{S}_2$ and $\mathcal{S}_3$. We denote the model of granular structures used in formal concept analysis as $(U, (Ext(L), \cap, \vee))$.

Example 3.2. Table 2, adopted from an example in [9], gives a formal context. The set of objects is $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, and the set of properties is $V = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$.

Table 2. A formal context

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
x_1	×	×					×		
x_2	×	×					×	×	
x_3	×	×	×				×	×	
x_4	×		×				×	×	×
x_5	×	×		×		×			
x_6	×	×	×	×		×			
x_7	×		×	×	×				
x_8	×		×	×		×			

From the discussion above, the family of all granules $Ext(L)$ can be obtained as:

$$Ext(L) = \{\emptyset, \{x_3\}, \{x_4\}, \{x_6\}, \{x_7\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_3, x_6\}, \{x_5, x_6\}, \{x_6, x_8\},$$
$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_6, x_7, x_8\}, \{x_5, x_6, x_8\}, \{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7, x_8\},$$
$$\{x_1, x_2, x_3, x_5, x_6\}, \{x_3, x_4, x_6, x_7, x_8\}, U\}.$$

It can be easily verified that $Ext(L)$ is a closure system, and it is closed under set intersection, but not closed under set union. For example, for $\{x_4\}, \{x_6\} \in Ext(L)$, we have $\{x_4\} \cup \{x_6\} = \{x_4, x_6\} \notin Ext(L)$.

3.3. Granular structure $(Q, (\mathcal{K}, \cup))$ in knowledge spaces

The theory of knowledge spaces represents a new paradigm in mathematical psychology [6, 7, 8]. It provides a systematic approach for knowledge assessment by considering a finite set of questions and a collection of subsets of questions called knowledge states. One may view a knowledge state as a granule and a knowledge space as a granular structure. The construction of a knowledge space is based on the notion of a surmise system [6, 7, 8].

Definition 3.3. Let Q be a finite set of questions. A surmise system on Q is a mapping σ that associates any question $q \in Q$ to a nonempty collection $\sigma(q)$ of subsets of Q satisfying the following three conditions:

$$\begin{aligned} &(1) \qquad X \in \sigma(q) \Longrightarrow q \in X, \\ &(2) \qquad (X \in \sigma(q), q' \in X) \Longrightarrow \exists X' \in \sigma(q')(X' \subseteq X), \\ &(3) \qquad X \in \sigma(q) \Longrightarrow \forall X' \in \sigma(q)(X' \not\subseteq X), \end{aligned}$$

where X is a subset of questions in $\sigma(q)$ called a clause for question q.

Semantically, a surmise system provides a list of prerequisite or background questions of a question. Each subset of questions in $\sigma(q)$ may be viewed as a possible history of the mastery of question q. That is, from the mastery of question q, one can surmise the mastery of all questions in one of the subsets in

$\sigma(q)$. Thus, the three conditions are reasonable. Condition (1) generalizes the reflexivity condition for a relation, while the condition (2) extends the notion of transitivity. Condition (3) requires that the clauses for question x are the maximal sets.

Based on a surmise system, one can construct a knowledge space by [6, 7, 8]:

$$\mathcal{K} = \{K \mid \forall q \in Q \ (q \in K \Longrightarrow \exists S \in \sigma(q) \ (S \subseteq K))\},$$

By the properties of a surmise system and construction of a knowledge space, it can be verified that $\emptyset \in \mathcal{K}$, $Q \in \mathcal{K}$, and $\mathcal{K}$ is closed under the set union. However, $\mathcal{K}$ may not be closed under set intersection. The model of granular structures in knowledge spaces is denoted by $(Q, (\mathcal{K}, \cup))$.

Example 3.3. Consider an example from [7]. Suppose $Q = \{a, b, c, d, e\}$. For a surmise system:

$$\begin{aligned}
&\sigma(a) = \{\{a\}\},\\
&\sigma(b) = \{\{b, d\}, \{a, b, c\}, \{b, c, e\}\},\\
&\sigma(c) = \{\{a, b, c\}, \{b, c, e\}\},\\
&\sigma(d) = \{\{b, d\}\},\\
&\sigma(e) = \{\{b, c, e\}\},
\end{aligned}$$

the induced knowledge space is:

$$\mathcal{K} = \{\emptyset,\ \{a\},\ \{b, d\},\ \{a, b, c\},\ \{b, c, e\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}, Q\}.$$

It is closed under set union, but not be closed under set intersection. For example, for $\{a, b, c\}, \{b, c, e\} \in \mathcal{K}$, we have $\{a, b, c\} \cap \{b, c, e\} = \{b, c\} \notin \mathcal{K}$.

4. Approximations with respect to granular structures

A granular structure usually consists of a subset of the power set. Semantically, subsets in the granular structures are constructed based available information and knowledge about the universe. They may be interpreted as, for example, definable subsets in rough set theory, extensions of concepts in formal concept analysis, and knowledge states in knowledge spaces. In other words, available information and knowledge enables to identify, represent and interpret these subsets. On the other hand, we cannot do so for subsets not in the granular structure; they must be approximated by subsets in the granular structure. Through their approximations, we can make inference about subsets not in the granular structure.

The theory of rough sets provides a method for constructing approximations [14, 25, 29, 36]. This section examines how to approximate these subsets that are not in a granular structure by subsets in the granular structure. We only study approximations in three models, a similar argument can be applied to other models.

4.1. Approximations in model $\mathcal{M}_7 = (U, (G, ^c, \cap, \cup))$

In rough set theory, the granular structure used is closed with respect to set complement, intersection and union. Given a subset $A \subseteq U$, Pawlak [18] suggests a method to approximate the sets from below and above by definable set in G.

Definition 4.1. Suppose $(G,^c,\cup,\cap)$ is a granular structure that is closed with respect to set complement, intersection and union. For a subset $A \subseteq U$, we can construct the following two families of subsets in G:

$$\mathbf{L}(A) = \{X \mid X \in G, X \subseteq A\},$$
$$\mathbf{H}(A) = \{X \mid X \in G, A \subseteq X\}.$$

That is, $\mathbf{L}(A)$ consists of subsets in G that are contained by A and $\mathbf{H}(A)$ consists of subsets in G that contain A. When approximating A from below and above, it is reasonable to choose maximal elements of $\mathbf{L}(A)$ and minimal elements of $\mathbf{H}(A)$, respectively. This results in the following definition.

Definition 4.2. For a subset A of the universe U, its lower and upper approximations are given by:

$$\underline{Apr}(A) = \{X \mid X \text{ is a maximal element of } \mathbf{L}(A)\},$$
$$\overline{Apr}(A) = \{X \mid X \text{ is a minimal element of } \mathbf{H}(A)\},$$

where a maximal M of $\mathbf{L}(A)$ is defined by the condition:

$$M \in \mathbf{L}(A) \wedge \forall M' \in \mathbf{L}(A)\ (M \subseteq M' \Longrightarrow M = M'),$$

and the minimal element N of $\mathbf{H}(A)$ is defined by the condition:

$$N \in \mathbf{H}(A) \wedge \forall N' \in \mathbf{H}(A)\ (N' \subseteq N \Longrightarrow N' = N).$$

Since G is closed under set intersection and union, there is a unique maximal element of $\mathbf{L}(A)$ and a unique minimal element of $\mathbf{H}(A)$. Thus, we can get the following definition [18, 36]:

Definition 4.3. For a subset $A \subseteq U$, a pair of approximations is given by:

$$\begin{aligned} \underline{apr}(A) &= \bigcup\{X \mid X \in \mathbf{L}(A), X \subseteq A\} \\ &= \bigcup\{X \mid X \in G, X \subseteq A\}, \\ \overline{apr}(A) &= \bigcap\{X \mid X \in \mathbf{H}(A), A \subseteq X\} \\ &= \bigcap\{X \mid X \in G, A \subseteq X\}. \end{aligned}$$

For $A, B \subseteq U$, it can be verified that the pair of approximations satisfies the following properties:

$$\begin{array}{ll} (0) & \underline{apr}(A) = (\overline{apr}(A^c))^c, \\ & \overline{apr}(A) = (\underline{apr}(A^c))^c, \\ (1) & \overline{apr}(A) \subseteq A \subseteq \underline{apr}(A), \\ (2) & A \in G \Longrightarrow \underline{apr}(A) = A = \overline{apr}(A), \\ (3) & A \subseteq B \Longrightarrow \underline{apr}(A) \subseteq \underline{apr}(B), \\ & A \subseteq B \Longrightarrow \overline{apr}(A) \subseteq \overline{apr}(B), \\ (4) & \underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B), \\ & \overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B). \end{array}$$

Additional properties can be found in rough set literature [18].

Example 4.1. Consider the granular structure G given by Example 3.1. The subset $A = \{x_1, x_2, x_3, x_4\}$ is an undefinable granule. We approximate it by a pair of definable sets of objects from below and above in G:

$$\underline{apr}(A) = \bigcup\{X \mid X \in G, X \subseteq A\} = \{x_1, x_2\} \cup \{x_3\} \cup \{x_1, x_2, x_3\} = \{x_1, x_2, x_3\},$$
$$\overline{apr}(A) = \bigcap\{X \mid X \in G, A \subseteq X\} = \{x_1, x_2, x_3, x_4, x_5\} = U.$$

The same definition can be used for defining approximations in Model $\mathcal{M}_6$. However, since G is not necessarily closed under set complement, we do not have property (0). Model $\mathcal{M}_6$ is a special case of lattice model $\mathcal{M}_3$. One can immediately apply the same formulation for defining approximations in a lattice. Many authors, for example, see Cattaneo [3] and Järvinen [11], studied generalized rough set approximations based on lattices.

4.2. Approximations in model $\mathcal{M}_4 = (U, (G, \cap))$

In model $\mathcal{M}_4$, a granular structure G contains the empty set $\emptyset$ and is closed under set intersection. In this case, there is a unique minimal element in $\mathbf{H}(A)$, but there may exist more than one maximal element in $\mathbf{L}(A)$. Hence, a pair of lower and upper approximations can be defined by two families of subsets from G, with the upper approximation to be a singleton set [35].

Definition 4.4. For a subset $A \subseteq U$, a pair of lower and upper approximations is given by:

$$\begin{aligned}
\underline{Apr}(A) &= \{X \mid X \text{ is a maximal element of } \mathbf{L}(A)\} \\
&= \{X \mid X \subseteq A, X \in \mathbf{L}(A), \forall Y \in \mathbf{L}(A)\ (X \subseteq Y \Longrightarrow X = Y)\} \\
&= \{X \mid X \in G, X \subseteq A, \forall Y \in G\ (X \subset Y \Longrightarrow Y \not\subseteq A)\} \\
\overline{Apr}(A) &= \{X \mid X \text{ is a minimal element of } \mathbf{H}(A)\} \\
&= \{\bigcap\{B \mid B \in \mathbf{H}(A), A \subseteq B\} \\
&= \{\bigcap\{B \mid B \in G, A \subseteq B\}\}.
\end{aligned}$$

Since G is closed under set intersection, the minimal element of $\mathbf{H}(A)$ is unique and is defined by $\bigcap\{B \mid B \in \mathbf{H}(A), A \subseteq B\}$. The set A is approximated from above by a single granule. On the other hand, there may not exist a unique maximal element of $\mathbf{L}(A)$. The set A may be approximated from below by several granules. Approximations in model $\mathcal{M}_4$ has been investigated in the context of formal concept analysis [35].

For $A, B \subseteq U$, the pair approximations satisfies the properties:

$$\begin{aligned}
&(1) \quad L \subseteq A \subseteq H, \text{ for } L \in \underline{Apr}(A), H \in \overline{Apr}(A); \\
&(2) \quad A \in G \Longrightarrow \underline{Apr}(A) = \{A\} = \overline{Apr}(A); \\
&(3) \quad A \subseteq B \Longrightarrow L_A \subseteq L_B, \text{ for } L_A \in \underline{Apr}(A), L_B \in \underline{Apr}(B), \\
&\qquad A \subseteq B \Longrightarrow H_A \subseteq H_B, \text{ for } H_A \in \overline{Apr}(A), H_B \in \overline{Apr}(B).
\end{aligned}$$

These properties may be viewed as counterparts of the properties in model $\mathcal{M}_7$

Example 4.2. With respect to the $\cap$-closed granular structure G in Example 3.2, the set $A = \{x_2, x_3, x_4, x_6\}$ is not a granule in G. We approximate it by a pair of families of definable sets of objects from below and above in G.

$$\underline{Apr}(A) = \{X \mid X \in G, X \subseteq A, \forall Y \in G\ (X \subset Y \Longrightarrow Y \not\subseteq A)\} = \{\{x_2, x_3, x_4\}, \{x_3, x_6\}\},$$
$$\overline{Apr}(A) = \{\bigcap\{X \mid X \in G, A \subseteq X\} = \{\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}\}.$$

There are two granules $\{x_2, x_3, x_4\}$ and $\{x_3, x_6\}$ for approximating A from below.

4.3. Approximations in model $\mathcal{M}_5 = (U, (G, \cup))$

As a dual granular structure of $\mathcal{M}_4$, $\mathcal{M}_5$ uses a granular structure that is closed under set union. For a subset $A \subseteq U$, there is a unique maximal element in **L**(A) and there may exist more than one minimal element in **H**(A). Accordingly, we have the following pair of approximations [25, 36].

Definition 4.5. For a subset $A \subseteq U$, a pair of approximations of A with respect to G is given by:

$$\begin{aligned}
\underline{Apr}(A) &= \{X \mid X \text{ is a maximal element of } \mathbf{L}(A)\} \\
&= \{\bigcup\{B \mid B \in \mathbf{L}(A), B \subseteq A\}\} \\
&= \{\bigcup\{B \mid B \in G, B \subseteq A\}\}, \\
\overline{Apr}(A) &= \{X \mid X \text{ is a minimal element of } \mathbf{H}(A)\} \\
&= \{X \mid A \subseteq X, X \in \mathbf{H}(A), \forall Y \in \mathbf{H}(A)\ (Y \subseteq X \Longrightarrow Y = X)\} \\
&= \{X \mid X \in G, X \subseteq A, \forall Y \in G\ (Y \subset X \Longrightarrow A \not\subseteq X)\}.
\end{aligned}$$

The set A is approximated from below by a single granule and may be approximated from above by several granules. Approximations in model $\mathcal{M}_5$ has been investigated in the context of knowledge spaces [25, 36]. Approximations in model $\mathcal{M}_5$ also satisfy the three properties given in model $\mathcal{M}_4$.

Example 4.3. For the a granular structure G given by Example 3.3, $A = \{b, c, d\}$ is not in G. It can approximated from below and above as:

$$\underline{Apr}(A) = \{\bigcup\{B \mid B \in G, B \subseteq A\}\} = \{\{b, d\}\},$$
$$\overline{Apr}(A) = \{X \mid X \in G, X \subseteq A, \forall Y \in G\ (Y \subset X \Longrightarrow A \not\subseteq X)\} = \{\{a, b, c, d\}, \{b, c, d, e\}\}.$$

There are two granules $\{a, b, c, d\}$ and $\{b, c, d, e\}$ for approximating A from above.

5. Conclusions

Granular computing emphasizes structured approaches to problem solving and information processing. Constructing a meaningful and practical granular structure is an important task in granular computing. In this paper, we propose and investigate a framework for studying set-theoretic models of granular structures. A granule is modelled by a subset of a universal set and a granular structure by a family of granules equipped with the standard set-inclusion relation. We examine three lattice-based models and three set-based models of granular structures, respectively. It is shown that rough set analysis, formal concept analysis and knowledge spaces indeed use one or a mixture of these models. The notion of approximations is also studied with respect to a granular structure.

The results in this paper provide a basis for unifying several data analysis theories. The proposed models of granular structures contribute to a better understanding of granular computing.

Acknowledgements

This work is partically supported by a Discovery Grant from NSERC Canada and No. 60970061 from the National Natural Science Foundation of China. The authors thank anonymous reviewers for their constructive comments.

References

[1] Bargiela, A., Pedrycz, W.: *Granular Computing: An Introduction*, Kluwer Academic Publishers, Boston, 2002.

[2] Caspard, N., Monjardet, B.: Some lattices of closure systems on a finite set, *Discrete Mathematics and Theoretical Computer Science*, **6**, 163-190, 2004.

[3] Cattaneo, G.: Abstract approximation spaces for rough theories, in: Polkowski, L. and Skowron, A. (Eds.), *Rough Sets in Knowledge Discovery 1: Methodology and Applications*, Physica-Verlag, Heidelberg, 59-98, 1998.

[4] Ciucci, D.: Approximation algebra and framework, *Fundamenta Informaticae*, **94**, 147-161, 2009.

[5] Ciucci, D.: Orthopairs: a simple and widely used way to model uncertainty, *Fundamenta Informaticae*, **108**, 287-304, 2011.

[6] Doignon, J.P., Falmagne, J.C.: Spaces for the assessment of knowledge, *International Journal of Man-Machine Studies*, **23**, 175-196, 1985.

[7] Doignon, J.P., Falmagne, J.C.: *Knowledge Spaces*, Springer-Verlag, Berlin, 1999.

[8] Falmagne, J.C., Koppen, M., Villano, M., Doignon, J.P., Johanessen, L.: Introduction to knowledge spaces: how to test and search them, *Psychological Review*, **97**, 201-224, 1990.

[9] Ganter, B., Wille, R.: *Formal Concept Analysis: Mathematical Foundations*, Springer-Verlag, Berlin, 1999.

[10] Hobbs, J.R.: Granularity, *Proceedings of the 9th International Joint Conference on Artificial Intelligence*, 432-435, 1985.

[11] Järvinen, J.: Lattice theory for rough sets, *LNCS Transactions on Rough Sets*, **6**, LNCS 4374, 400-498, 2007.

[12] Keet, C.M.: A taxonomy of types of granularity, *Proceeding of the 2006 IEEE International Conference on Granular Computing*, 106-111, 2006.

[13] Keet, C.M.: *A Formal Theory of Granularity*, PhD Thesis, KRDB Research Centre, Faculty of Computer Science, Free University of Bozen-Bolzano, Italy (2008), http://www.meteck.org/files/AFormalTheoryOfGranularity_CMK08.pdf (accessed June 8, 2008)

[14] Li, T.J., Jing, Y.L.: Rough set approximations on granular structures and feature characterizations, in: Zhang, Y.C., Alfredo, C., Ma, J.H., Chung, K.I., Arslan, T. and Song, X.F. (Eds.). *Database Theory and Application, Bio-Science and Bio-Technology*, Springer, Berlin, 79-88, 2010.

[15] Miao, D.Q., Fan, S.D.: The calculation of knowledge granulation and its application, *System Engeering-Theory and Practice*, **22**, 48-56, 2002.

[16] Miao, D Q., Wang, G.Y., Liu, Q., Lin, T.Y., Yao, Y.Y. (Eds.): *Granular Computing: Past, Present and Future*, Science Press, Beijing, 2007.

[17] Nguyen, H.S., Skowron, A., Stepaniuk, J.: Granular computing: a rough set approach, *Computational Intelligence*, **17**, 514-544, 2001.

[18] Pawlak, Z.: Rough sets, *International Journal of Computer and Information Sciences*, **11**, 341-356, 1982.

[19] Pawlak, Z.: *Rough Sets-Theoretical Aspects of Reasoning About Data*, Kluwer Academic Publishers, Boston, 1991.

[20] Pawlak, Z., Skowron, A.: Rough sets: some extensions, *Information Sciences*, **177**, 28-40, 2007.

[21] Pedrycz, W., Skowron, A., Kreinovich, V. (Eds.): *Handbook of Granular Computing*, Wiley-Interscience, New York, 2008.

[22] Shiu, L.P., Sin, C.Y.: Top-down, middle-out, and bottom-up processes: a cognitive perspective of teaching and learning economics, *International Review of Economics Education*, **5**, 60-72, 2006.

[23] Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concepts, in: Rival I. (Ed.), *Ordered sets*, Reidel, Boston, 445-470, 1982.

[24] Wille, R.: Concept lattices and conceptual knowledge systems, *Computers Mathematics with Applications*, **23**, 493-515, 1992.

[25] Xu, F.F., Yao, Y.Y., Miao, D.Q.: Rough set approximations in formal concept analysis and knowledge spaces, *Proceedings of 17th International Symposium of Foundations of Intelligent Systems*, LNCS(LNAI) 4994, 319-328, 2008.

[26] Yao, J.T.: A ten-year review of granular computing, *Proceedings of 2007 IEEE Internationational Conference on Granular Computing*, 734-739, 2007.

[27] Yao, J.T. Recent developments in granular computing: a bibliometrics study, *Proceedings of 2008 IEEE International Conference on Granular Computing*, 74-79, 2008.

[28] Yao, Y.Y.: On generalizing Pawlak approximation operators, *Proceedings of the 1st International Conference on Rough Sets and Current Trends in Computing*, LNCS(LNAI) 1424, 298-307, 1998.

[29] Yao, Y.Y.: On generalizing rough set theory, *Proceedings of the 9th International Conference on Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing*, LNCS(LNAI) 2639, 44-51, 2003.

[30] Yao, Y.Y.: A comparative study of formal concept analysis and rough set theory in data analysis, *Proceedings of 4th International Conference of Rough Sets and Current Trends in Computing*, LNCS(LNAI) 3066, 59-68, 2004.

[31] Yao, Y.Y.: Perspectives of granular computing, *Proceedings of 2005 IEEE International Conference on Granular Computing*, 85-90, 2005.

[32] Yao, Y.Y.: Three perspectives of granular computing, *Journal of Nanchang Institute of Technology*, **25**, 16-21, 2006.

[33] Yao, Y.Y.: Granular computing: past, present and future, *Proceedings of 2008 IEEE International Conference on Granular Computing*, 80-85, 2008.

[34] Yao, Y.Y.: Interpreting concept learning in cognitive informatics and granular computing, *IEEE Transactions on Systems, Man, and Cybernetics (Part B)*, **39**, 855-866, 2009.

[35] Yao, Y.Y., Chen, Y.H.: Rough set approximations in formal concept analysis, *LNCS Transactions on Rough Sets*, V, LNCS 4100, 285-305, 2006.

[36] Yao, Y.Y., Miao, D.Q., Xu, F.F.: Granular structures and approximations in rough sets and knowledge spaces, in: Abraham, A., Falcón, R. and Bello, R. (Eds.), *Rough Set Theory: A True Landmark in Data Analysis*, Springer, Berlin, 71-84, 2009.

[37] Yao, Y.Y., Miao, D.Q., Zhang, N., Xu, F.F.: Set-theoretic models of granular structures, *Proceedings of the 5th International Conference on Rough Set and Knowledge Technology*, LNCS(LNAI) 6401, 94-101, 2010.

[38] Yao, Y.Y., Zhou, B.: A logic language of granular computing, *Proceedings 6th IEEE International Conference on Cognitive Informatics*, 178-185, 2006.

[39] Zadeh, L.A.: Towards a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, **90**, 111-127, 1997.