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Two Views of the Theory of Rough Sets in Finite Universes

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Abstract

This paper presents and compares two views of the theory of rough sets. The operator-oriented view interprets rough set theory as an extension of set theory with two additional unary operators. Under such a view, lower and upper approximations are related to the interior and closure operators in topological spaces, the necessity and possibility operators in modal logic, and lower and upper approximations in interval structures. The set-oriented view focuses on the interpretation and characterization of members of rough sets. Iwinski type rough sets are formed by pairs of definable (composed) sets, which are related to the notion of interval sets. Pawlak type rough sets are defined based on equivalence classes of an equivalence relation on the power set. The relation is defined by the lower and upper approximations. In both cases, rough sets may be interpreted, or related to, families of subsets of the universe, i.e., elements of a rough set are subsets of the universe. Alternatively, rough sets may be interpreted using elements of the universe based on the notion of rough membership functions. Both operator-oriented and set-oriented views are useful in the understanding and application of the theory of rough sets.

Keywords: approximation operators, fuzzy sets, interval sets, interval structures, modal logic, rough membership functions, rough sets, topological spaces, uncertain reasoning

1 Introduction

The theory of rough sets is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information [38,39,43].

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The introduction of the notion of rough sets is motivated by the practical needs in classification and concept formation with incomplete information [44]. It is different from, and complementary to, other generalizations, such as fuzzy sets and multisets [10,19,44,61]. There has been a fast growing interest in this new emerging theory. The successful applications of rough set models in a variety of problems have amply demonstrated their usefulness and versatility [21,25,42,53,70].

In a rough set model, elements of the universe are described in the context of available information (knowledge) about them. For example, in a medical expert system patients are normally described by their symptoms. In a pattern recognition system, objects may be described by their features. When two distinct objects are described by the same description, they will be perceived as the same or being indistinguishable. This may be formally described by an equivalence relation, i.e., a reflexive, symmetric and transitive relation. Each equivalence class consists of all these elements that are indistinguishable. Given an arbitrary subset of the universe, one may not be able to describe it precisely using the available information, i.e., equivalence classes of an equivalence relation on the universe. Instead, one can form a pair of approximations. The lower approximation is the union of all the equivalent classes which are subsets of the set, and the upper approximation is the union of all the equivalent classes which have a non-empty intersection with the set. The set lies within its lower and upper approximations. In this formulation, the notion of binary relations, representing relationships between the elements of the universe, is the primitive concept. The theory of rough sets is formulated using the information, expressed in terms of binary relations, about elements of the universe.

Many different proposals have been proposed for generalizing and interpreting rough sets [15,58,63,68]. Extensive research has been carried out to compare the theory of rough sets with other theories of uncertainty, such as fuzzy sets [6,10,19,20,57,59,61], modal logic [23,27,33,35,67], conditional events [29,30], Dempster-Shafer theory of evidence [51,52], and approximation theory [24]. The results of these studies enhance our understanding of rough sets, and provide new problem solving techniques in many sub-areas of artificial intelligence. On the other hand, these studies also show that some proposed generalizations are very different from each other. There exist many different interpretations of the notion of rough sets.

In this paper, we argue that two related and distinct views, the operator-oriented view and the set-oriented view, may be used for the interpretation of the theory of rough sets. A common underlying concept for both views is the notion of approximation spaces and the induced lower and upper approximations. The difference between these two views lies in the ways in which the lower and upper approximations are interpreted. The operator-oriented view interprets approximations as a pair of unary operators on the power set of the universe. That is, the theory of rough sets is an extension of set theory with two approximation operators. With the set-oriented view, lower and upper approximations are used to define the notion of rough sets. There are two ways for achieving this task, one uses subsets of the universe, and the other uses elements of the universe. Each of the proposed views captures different and important aspects of the concept of rough sets. Using the proposed two views, we present a review of existing interpretations of rough sets, and investigate the connections between the theory of rough sets and other theories of uncertainty. The operator-oriented view is related to topological space, modal logic, Boolean algebra with added operators, and interval structures. The set-oriented view is related to interval sets and fuzzy sets.

The main objective of this paper is to give a synthesis of many different interpretations of rough sets. Although new results are presented, a major part the paper is devoted to revealing interconnections between different interpretations of rough sets, and between the theory of rough sets and other theories of uncertainty. The results of the present study may provide a general framework for future research.

Appendix contains the proofs of several theorems that do not immediately follow from the discussion in the text. It is important to note that in this paper we only consider finite universes. Results obtained for finite universes may not necessarily hold if the universe is infinite.

2 Approximation Spaces

The notion of approximation spaces is one of the fundamental concepts in the theory of rough sets. This section presents a review of the Pawlak approximation space constructed from an equivalence relation and its generalization using any binary relations.

2.1 Pawlak Approximation Space

Let U denote a finite and non-empty set called the universe. Let $\Re \subseteq U \times U$ be an equivalence relation on U. The pair $apr = (U, \Re)$ is called a Pawlak approximation space. The equivalence relation $\Re$ partitions the set U into disjoint subsets. Let $U/\Re$ denote the quotient set consisting of equivalence classes of $\Re$. The empty set $\emptyset$ and the elements of $U/\Re$ are called elementary sets. A finite union of elementary sets, i.e., the union of one or more elementary sets, is called a composed set [19]. The family of all composed sets is denoted by Com(apr). It is a subalgebra of the Boolean algebra 2^U formed by the power set of U. A set which is a union of elementary sets is called a definable set [19]. The family of all definable sets is denoted by Def(apr). For a finite universe, the family of definable sets is the same as the family of composed sets. A Pawlak approximation space defines uniquely a topological space $(U, \mathrm{Def}(apr))$, in which Def(apr) is the family of all open and closed sets [38].

Given an arbitrary set $A \subseteq U$, in general it may not be possible to describe A precisely in $apr = (U, \Re)$. One may characterize A by a pair of lower and upper approximations. The following definitions summarize some of the proposals.

Definition 1 *Let $\Re$ be an equivalence relation on a universe U, $[x]_\Re$ the equivalence class containing x, and* Def(apr) *the family of all definable sets. For any set $A \subseteq U$, the lower approximation $\underline{apr}_\Re(A)$ and the upper approximation $\overline{apr}_\Re(A)$ are defined by:*

(i). $\underline{apr}_\Re(A)$ is the greatest definable set contained in A,
$\overline{apr}_\Re(A)$ is the least definable set containing A;

(ii). $\underline{apr}_\Re(A) = \bigcup\{X \mid X \subseteq A, X \in \mathrm{Def}(apr)\}$,
$\overline{apr}_\Re(A) = \bigcap\{X \mid A \subseteq X, X \in \mathrm{Def}(apr)\}$;

(iii). $\underline{apr}_\Re(A) = \{x \in U \mid [x]_\Re \subseteq A\}$,
$\overline{apr}_\Re(A) = \{x \in U \mid [x]_\Re \cap A \neq \emptyset\}$,

(iv). $\underline{apr}_\Re(A) = \{x \in U \mid \text{for all } y \in U,\ x\Re y \text{ implies } y \in A\}$,
$\overline{apr}_\Re(A) = \{x \in U \mid \text{there exists a } y \in U \text{ such that } x\Re y \text{ and } y \in A\}$;

(v). $\underline{apr}_\Re(A) = \bigcup\{[x]_\Re \mid [x]_\Re \in U/\Re, [x]_\Re \subseteq A\}$,
$\overline{apr}_\Re(A) = \bigcup\{[x]_\Re \mid [x]_\Re \in U/\Re, [x]_\Re \cap A \neq \emptyset\}$;

(vi). $\underline{Apr}_\Re(A) = \{[x]_\Re \mid [x]_\Re \in U/\Re, [x]_\Re \subseteq A\}$,
$\overline{Apr}_\Re(A) = \{[x]_\Re \mid [x]_\Re \in U/\Re, [x]_\Re \cap A \neq \emptyset\}$.

It is important to note that the equivalence class $[x]_\Re$ containing x plays dual roles. It is a subset of U if considered in relation to the universe, and an element of $U/\Re$ if considered in relation to the quotient set. Lin [19], following Dubois and Prade [10], explicitly used $[x]_\Re$ for representing a subset of U and $Name([x]_\Re)$ for representing an element of $U/\Re$. For simplicity, in this paper we will used the same symbol $[x]_\Re$. Its particular role can be identified from the context.

Definitions (i) to (vi) have been studied by many authors [1,3,12,16,17,31,32,36,38,39,46,47,63]. Except the last one, they are indeed equivalent definitions. Each of them captures different aspects of approximations and offers various interpretations. The set Def(apr) is the family of all open and closed sets in the topological space $(U, \mathrm{Def}(apr))$. Using definitions (i) and (ii), approximations can be viewed as the interior and closure operators in a topological space. The power set 2^U is a Boolean algebra and $\mathrm{Def}(apr)$ is a sub-Boolean algebra. Definitions (i) and (ii) may be generalized to any Boolean algebra. An equivalence relation defines uniquely a partition, and vice versa. Thus, Definition (iv) is essentially a restatement of definition (iii). It explicitly uses the equivalence relation, instead of the induced equivalence classes, which offers a straightforward way to generalize rough set model using different types of binary relations [58,63]. Definition (v) clearly states the relationships between approximations and elementary sets. The lower approximation is the union of all the elementary sets which are subsets of A, and the upper approximation is the union of all the elementary sets which have a non-empty intersection with A. Definition (vi) defines approximations in terms of the elements of the quotient set $U/\Re$. They can be transformed into approximations consisting of elements of U:

$$\begin{aligned} \underline{apr}_{\Re}(A) &= \bigcup\{X \mid X \in \underline{Apr}_{\Re}(A)\}, \\ \overline{apr}_{\Re}(A) &= \bigcup\{X \mid X \in \overline{Apr}_{\Re}(A)\}. \end{aligned} \tag{1}$$

They may be considered as a special case of interval structures [56].

The lower and upper approximations satisfy the following properties: for subsets $A, B \subseteq U$,

(L1) $\underline{apr}_{\Re}(A) = \ \sim \overline{apr}_{\Re}(\sim A)$,
(L2) $\underline{apr}_{\Re}(U) = U$,
(L3) $\underline{apr}_{\Re}(A \cap B) = \underline{apr}_{\Re}(A) \cap \underline{apr}_{\Re}(B)$,
(L4) $\underline{apr}_{\Re}(A \cup B) \supseteq \underline{apr}_{\Re}(A) \cup \underline{apr}_{\Re}(B)$,
(L5) $A \subseteq B \Longrightarrow \underline{apr}_{\Re}(A) \subseteq \underline{apr}_{\Re}(B)$,
(L6) $\underline{apr}_{\Re}(\emptyset) = \emptyset$,
(L7) $\underline{apr}_{\Re}(A) \subseteq A$,
(L8) $A \subseteq \underline{apr}_{\Re}(\overline{apr}_{\Re}(A))$,
(L9) $\underline{apr}_{\Re}(A) \subseteq \underline{apr}_{\Re}(\underline{apr}_{\Re}(A))$,
(L10) $\overline{apr}_{\Re}(A) \subseteq \underline{apr}_{\Re}(\overline{apr}_{\Re}(A))$,

(U1) $\overline{apr}_{\Re}(A) = \ \sim \underline{apr}_{\Re}(\sim A)$,
(U2) $\overline{apr}_{\Re}(\emptyset) = \emptyset$,
(U3) $\overline{apr}_{\Re}(A \cup B) = \overline{apr}_{\Re}(A) \cup \overline{apr}_{\Re}(B)$,
(U4) $\overline{apr}_{\Re}(A \cap B) \subseteq \overline{apr}_{\Re}(A) \cap \overline{apr}_{\Re}(B)$,
(U5) $A \subseteq B \Longrightarrow \overline{apr}_{\Re}(A) \subseteq \overline{apr}_{\Re}(B)$,
(U6) $\overline{apr}_{\Re}(U) = U$,

(U7) $A \subseteq \overline{apr}_{\Re}(A)$,
(U8) $\overline{apr}_{\Re}(\underline{apr}_{\Re}(A)) \subseteq A$,
(U9) $\overline{apr}_{\Re}(\overline{apr}_{\Re}(A)) \subseteq \overline{apr}_{\Re}(A)$,
(U10) $\overline{apr}_{\Re}(\underline{apr}_{\Re}(A)) \subseteq \underline{apr}_{\Re}(A)$,

(K) $\underline{apr}_{\Re}(\sim A \cup B) \subseteq \ \sim \underline{apr}_{\Re}(A) \cup \underline{apr}_{\Re}(B)$,
(LU) $\underline{apr}_{\Re}(A) \subseteq \overline{apr}_{\Re}(A)$,

where $\sim A = U - A$ denotes the set complement of A. Properties (L1) and (U1) state that two approximations are dual to each other. Hence, properties with the same number may be regarded as dual properties. Properties (L9), (L10), (U9) and (U10) are expressed in terms of set inclusion. The standard version using set equality can be derived from (L1)-(L10) and (U1)-(U10). For example, it follows from (L7) and (L9) that $\underline{apr}_{\Re}(A) = \underline{apr}_{\Re}(\underline{apr}_{\Re}(A))$. It should also be noted that these properties are not independent.

With respect to any subset $A \subseteq U$, the universe can be divided into three disjoint regions using the lower and upper approximations:

$$\begin{aligned} &\mathrm{POS}(A) = \underline{apr}_{\Re}(A), \\ &\mathrm{NEG}(A) = \mathrm{POS}(\sim A) = U - \overline{apr}_{\Re}(A), \\ &\mathrm{BND}(A) = \overline{apr}_{\Re}(A) - \underline{apr}_{\Re}(A). \end{aligned} \quad (2)$$

An element of the positive region POS(A) definitely belongs to A, an element of the negative region NEG(A) definitely does not belong to A, and an element of the boundary region BND(A) only possibly belongs to A.

2.2 Generalized Approximation Spaces

Suppose $\Re$ is an arbitrary binary relation on U. The pair $apr = (U, \Re)$ is called a generalized approximation space or simply an approximation space. With respect to $\Re$, we can define a mapping $r : U \longrightarrow 2^U$:

$$r(x) = \{y \mid x\Re y\}, \quad (3)$$

by collecting all $\Re$-related elements of x. It is an equivalent, and sometimes more convenient, representation of a binary relation. If $\Re$ is indeed an equivalence relation, $r(x)$ is the equivalence class containing x. In generalizing definitions (iii)-(vi), one may use $r(x)$ in the place of the equivalence class $[x]_{\Re}$. By using different types of binary relations, one obtains distinct classes of approximation spaces [16,58,63,68].

In some studies, approximation spaces are defined using a covering by exploiting the fact that a covering is a generalization of a partition. One can generalize definitions (iii)-(vi), by substituting quivalence classes with elements of a covering and a partition with a covering, to derive approximation spaces [68]. Given a reflexive binary relation $\Re$, one may define a covering of the universe. For example, if $\Re$ is a reflexive relation, the set $\{r(x) \mid x \in U\}$ form a covering of U. Given a covering a U, one may construct a binary relation. There does not exist an one-to-one relationship between coverings and binary relations. In contrast to Pawlak approximation space, definitions based on a cover and a particular choice of binary relation may not necessarily consistent. This offers many avenues of possible extensions. For example, definitions (iii) and (iv) are equivalent, while definitions (v) and (vi) are consistent. They define two distinct types of approximations [47,58]. For clarity and simplicity, in this study we construct approximation spaces based on binary relations and definition (iii).

Definition 2 *Let $\Re$ be an arbitrary binary relation on a universe U and $r(x)$ the set of $\Re$-related elements of x. For any set $A \subseteq U$, a pair of lower and upper approximations, $\underline{apr}_{\Re}(A)$ and $\overline{apr}_{\Re}(A)$, are defined by:*

$$\begin{aligned} \underline{apr}_{\Re}(A) &= \{x \in U \mid r(x) \subseteq A\}, \\ \overline{apr}_{\Re}(A) &= \{x \in U \mid r(x) \cap A \neq \emptyset\}. \end{aligned} \tag{4}$$

By definition, we have $x \in \overline{apr}(\{y\}) \iff r(x) \cap \{y\} \neq \emptyset \iff y \in r(x)$. The binary relation can be reconstructed from upper approximations of singleton subsets of U:

$$r(x) = \{y \mid x \in \overline{apr}_{\Re}(\{y\})\}. \tag{5}$$

The lower and upper approximations satisfy properties (L1)-(L5), (U1)-(U5) and (K). In general, they do not satisfy other properties. Yao *et al.* [63] analyzed a number of different types of approximation spaces based on properties of the binary relation. By imposing additional properties on the binary relation, one can construct more specific approximation spaces in which lower and upper approximations have additional properties. If the binary relation is reflexive and symmetric, i.e., $\Re$ is a tolerance relation, one obtains the approximation space proposed by Zakowski [47,58,68]. Additional properties (L6)-(L8), (U6)-(U8) and (LU) hold. If $\Re$ is reflexive and transitive, one derives another approximation space which is a topological space [16,47,58]. Additional properties (L6), (L7), (L9), (U6), (U7), (U9) and (LU) hold.

3 Operator-oriented View

In this section, we present an operator-oriented view of rough sets by introducing the notion of rough set algebras. Relationships between rough set algebras and other mathematical structures are investigated.

3.1 Rough Set Algebras

Given an approximation space $apr = (U, \Re)$, it defines a pair of lower and upper approximations $\underline{apr}_{\Re}$ and $\overline{apr}_{\Re}$. By viewing them as a pair of dual unary operators on 2^U, one obtains a system $R = (2^U, \cap, \cup, \sim, \underline{apr}_{\Re}, \overline{apr}_{\Re})$. We call R a rough set algebra defined by the approximation space $apr = (U, \Re)$. It extends the standard set algebra $(2^U, \cap, \cup, \sim)$ by adding two set-theoretic operators [3,16,22,58,63]. Properties of rough set algebra are determined by the unary operators $\underline{apr}_{\Re}$ and $\overline{apr}_{\Re}$, which are determined by properties of the binary relation $\Re$.

Lin and Liu [22] considered a reverse process for defining rough set algebras. Instead of starting from a binary relation, they took an axiomatic approach by stating explicitly properties on approximation operators. One of their main objectives is to investigate the conditions on approximation operators so that they are equivalent to the ones defined by a binary relation. However, their formulation is set in the context of Pawlak approximation spaces. With respect to generalized approximation spaces, we have the following theorem stating the conditions on the approximation operators.

Theorem 3 *Suppose $L, H : 2^U \longrightarrow 2^U$ is a pair of dual operators, i.e., for all $A \subseteq U$, $L(A) = \sim H(\sim A)$. If H satisfies the following axioms:*

(c1) $H(\emptyset) = \emptyset$,
(c2) $H(A \cup B) = H(A) \cup H(B)$,

there exists a binary relation $\Re$ on U such that for all $A \subseteq U$, $L(A) = \underline{apr}_{\Re}(A)$ and $H(A) = \overline{apr}_{\Re}(A)$.

A constructive proof is given in the Appendix, in which we explicitly define a binary relation and show that the binary relation indeed produces the same approximation operators as L and H. An important implication of this theorem is that one can define the notion of rough set algebras by a pair of dual unary set-theoretic operators using axioms (c1) and (c2). If additional axioms are used, more specific rough set algebras will be derived. Examples of such algebras will be introduced in the following subsections.

3.2 Interior and Closure Operators in Topological Spaces

A topological space can be described by using a pair of interior and closure operators [49]. There may exist some relationships between a topological space and a rough set algebra, as the latter can also be described by a pair of operators in a similar manner. In fact, the lower and upper approximation operators in a Pawlak approximation space can be interpreted as a pair of interior and closure operators in the topological space $(U, \mathrm{Def}(apr))$. A rough set model may therefore be considered as a method for constructing a topological space using a binary relation on the universe. In a reverse process, one can generalize the notion of rough sets based on topological spaces by using definitions (i) and (ii). Definable sets are substituted by open sets in defining the lower approximation, and by closed set in defining the upper approximation [32,55].

In general, a pair of interior and closure operators characterized by Kuratowski axioms may not satisfy all properties of the Pawlak rough set algebra. For instance, property (L10) may not hold. One may use a weaker binary relation and still keep the interpretation of approximation operators as interior and closure operators. The following theorem states that a reflexive and transitive relation is sufficient for the approximation operators to be interior and closure operators [16].

Theorem 4 *Suppose $\Re$ is a reflexive and transitive relation on U. The pair of lower and upper approximations is a pair of interior and closure operators satisfying Kuratowski axioms.*

A rough set algebra constructed from a reflexive and transitive relation is referred to as a topological rough set algebra. Based on the axioms of closure operators in topological spaces, we may state the axioms on approximation operators in rough set algebras. The following two theorems state that a Pawlak rough algebra can be defined by five axioms, while a topological rough algebra can be defined by four axioms.

Theorem 5 *Suppose $L, H : 2^U \longrightarrow 2^U$ is a pair of dual operators. If H satisfies axioms (c1), (c2) and:*

(c3) $A \subseteq H(A)$,
(c4) $H(H(A)) = H(A)$,
(c5) $A \subseteq \ \sim H(\sim H(A))$,

there exists an equivalence relation $\Re$ on U such that for all $A \subseteq U$, $L(A) = \underline{apr}_{\Re}(A)$ and $H(A) = \overline{apr}_{\Re}(A)$.

Theorem 6 *Suppose* $L, H : 2^U \longrightarrow 2^U$ *is a pair of dual operators. If* H *satisfies axioms (c1)-(c4), there exists a reflexive and transitive relation* $\Re$ *on* U *such that for all* $A \subseteq U$, $L(A) = \underline{apr}_{\Re}(A)$ *and* $H(A) = \overline{apr}_{\Re}(A)$.

Axioms (c1)-(c4) are in fact Kuratowski axioms of closure operators [49]. Therefore, approximation operators in other types of rough set algebras may not be viewed as interior and closure operators as defined by Kuratowski axioms. In the above theorems, we used axioms on the operator H. Alternatively, one may also use the following dual axioms of (c1)-(c5):

(i1) $L(U) = U$,
(i2) $L(A \cap B) = L(A) \cap L(B)$,
(i3) $L(A) \subseteq A$,
(i4) $L(L(A)) = L(A)$,
(i5) $\sim L(\sim L(A)) \subseteq A$.

The operators L and H uniquely determine each other by the relationships $H(A) = \ \sim L(\sim (A)$ and $L(A) = \ \sim H(\sim H)$. Thus, axioms (c5) and (i5) may be more conveniently expressed as $A \subseteq L(H(A))$ and $H(L(A)) \subseteq A$.

Lin and Liu [22] proposed an interpretation of approximation operators in terms of interior and closure operators in Frechet topology. Let $n(x) \subseteq U$ denote a neighborhood of x. A neighborhood system $N(x)$ of x is a non-empty family of neighborhoods of x. The family of all such neighborhood systems determines a Frechet topological space (or Frechet space for short). A topological space is a Frechet space, but the reverse is not true [22]. Thus, Frechet space provides a more general framework in which rough set algebra may be interpreted. Based on neighborhood systems, a pair of approximation operators are defined by:

$$\begin{aligned} \underline{F}(A) &= \{x \mid \text{ there exists an } n(x) \in N(x) \text{ such that } n(x) \subseteq A\}, \\ \overline{F}(A) &= \{x \mid \text{ for all } n(x) \in N(x),\ n(x) \cap A \neq \emptyset\}. \end{aligned} \tag{6}$$

They are referred to as interior and closure operators of neighborhood systems. In our formulation, the set $r(x)$ may be considered as the only one neighborhood of x. This produces a special type of Frechet topology. Comparing equations (4) and (6), it is clearly that they are equivalent. Therefore, for a serial relation, i.e., for every element $x \in U$ there exists at least an element $y \in U$ such that $x \Re y$, approximation operators can be interpreted as interior and closure operators in a Frechet topology.

3.3 Necessity and Possibility Operators in Modal Logics

Consider a problem of reasoning about a particular situation. Typically, we have a fixed finite and non-empty set of primitive propositions Φ, which can be thought of as corresponding to basic events [11]. The set $L(\Phi)$ of propositional modal formulas is the closure of Φ under negation ($\neg$), conjunction ($\wedge$), and necessity ($\Box$). For convenience, we assume that there are two special formulas $\top$ and $\bot$. Other connectives such as the disjunction ($\vee$), implication ($\rightarrow$), equivalence ($\leftrightarrow$), and possibility ($\Diamond$) can be defined in terms of negation, conjunction and necessity. Let W be a non-empty set of possible worlds and $\Re$ a binary relation, called accessibility relation, on W. The pair $(W, \Re)$ is referred to as a frame. An interpretation in $(W, \Re)$ is a valuation function $v : W \times \Phi \longrightarrow \{true,\ false\}$, which assigns a truth value for each proposition with respect to each particular world w. If $v(w, p) = true$, we say that the proposition p is true in the interpretation v at the world w, written $w \models_v p$. The valuation function can be extended to formulas in $L(\Phi)$ recursively as follows:

(0) for $p \in \Phi, w \models_{v^*} p$ iff $w \models_v p$,
(1) not $w \models_{v^*} \bot, \qquad w \models_{v^*} \top$,
(2) $w \models_{v^*} (p \wedge q)$ iff $w \models_{v^*} p$ and $w \models_{v^*} q$,
(3) $w \models_{v^*} (p \vee q)$ iff $w \models_{v^*} p$ or $w \models_{v^*} q$, or both
(4) $w \models_{v^*} (p \rightarrow q)$ iff not $w \models_{v^*} p$ or $w \models_{v^*} q$, or both
(5) $w \models_{v^*} \neg p$ iff not $w \models_{v^*} p$,
(6) $w \models_{v^*} \Box p$ iff for all $w' \in W$, $w \Re w'$ implies $w' \models_{v^*} p$,
(7) $w \models_{v^*} \Diamond p$ iff there exists a $w' \in W$ such that $w \Re w'$ and $w' \models_{v^*} p$.

When the extended valuation function v^* is clear from context, we drop it by simply writing $w \models p$.

With a valuation function, we can characterize a proposition by the set of possible worlds in which the proposition is true. In other words, we can define a mapping $T : L(\Phi) \longrightarrow 2^W$ as follows:

$$T(p) = \{w \in W \mid w \models p\}. \tag{7}$$

The set $T(p)$ is referred to as the truth set of the proposition [7]. It is also called the incidence of p and the mapping T is called an incidence mapping [4]. It can be easily verified that the logical connectives can be interpreted using set-theoretic operators:

(s0) $T(\bot) = \emptyset, \qquad T(\top) = W$,

(s1) $T(p \wedge q) = T(p) \cap T(q)$,
(s2) $T(p \vee q) = T(p) \cup T(q)$,
(s3) $T(p \rightarrow q) = \ \sim T(p) \cup T(q)$,
(s4) $T(\neg p) = \ \sim T(p)$,
(s5) $T(\Box p) = \underline{apr}_{\Re}(T(p))$,
(s6) $T(\Diamond p) = \overline{apr}_{\Re}(T(p))$.

Such an interpretation was also used by Chakraborty and Banerjee [5], Orlowska [34,35], and Pawlak [41]. Using the truth set representation, a relationship between approximation operators and modal operators can be established.

Theorem 7 *Suppose $R = (2^W, \cap, \cup, \sim, \underline{apr}_{\Re}, \overline{apr}_{\Re})$ is a rough set algebra defined by an approximation space $apr = (W, \Re)$. Suppose a modal logic system $M = (L(\Phi), \wedge, \vee, \neg, \Box, \Diamond)$ is defined with respect to the frame $(W, \Re)$. The mapping T is a homomorphism from M to R.*

Properties of the approximation operators are related to the axioms of the modal operators. For example, the axioms corresponding to (K), (LU), and (L7)-(L10) are given by:

(K)	$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box p)$,
(D)	$\Box p \rightarrow \Diamond p$,
(T)	$\Box p \rightarrow p$,
(B)	$p \rightarrow \Box\Diamond p$,
(4)	$\Box p \rightarrow \Box\Box p$,
(5)	$\Diamond p \rightarrow \Box\Diamond p$.

By combining these axioms, one can define and classify additional rough set algebras [64]. Following the convention of labeling distinct modal logic system [7], one may label different types of rough set algebra [64]. These axioms are not independent. In labeling various rough set algebras, one only needs to list the independent axioms. For example, the Pawlak rough set algebra is labled by KT5, while the topological rough set algebra is labeled by KT4. Other rough set algebras have been examined by Yao and Lin [64].

In the study of modal logic systems, the notion of Boolean algebra has been extended by adding new operators [14,18,49,54]. Based on the properties of such operators, many classes of extended Boolean algebras can be constructed. For example, the topological Boolean algebra is characterized by axioms similar to (c1)-(c4) of topological rough set algebra [49]. It is important to realize that such a study of rough sets is indeed parallel to the algebraic approach to modal logic.

3.4 Lower and Upper Approximations in Interval Structures

In definition (vi), the lower and upper approximation operators are defined as mappings between two distinct sets, namely, from the power set 2^U to the power set $2^{U/\Re}$. The notion of interval structures may be considered as a generalization of this definition [56].

Let U and W denote two finite universes of interest and C a binary relation which is a subset of the Cartesian product $W \times U$. The relation C is called a compatibility relation, reflecting a plausible physical interpretation of the binary relation [50,56]. We call the triplet $Apr = (W, U, C)$ an approximation space. Without loss of generality, we may assume that for any $w \in W$ there exists a $x \in U$ with $w\ C\ x$, and vice versa. This assumption is related to the condition of serial binary relation if W and U are chosen to be the same. Similar to equation (3), the relation C can be equivalently defined by a mapping $r_C : W \longrightarrow 2^U$:

$$r_C(w) = \{x \in U \mid w\ C\ x\}. \tag{8}$$

That is, $r_C(w)$ is a subset of U consisting of all elements compatible with w. A binary relation on the same universe is a special case of a binary relation on two universes. By extending definition (iii), for each subset $A \subseteq U$, we define a pair of lower and approximations using the elements of W:

$$\begin{aligned} \underline{Apr}_C(A) &= \{w \in W \mid r_C(w) \subseteq A\}, \\ \overline{Apr}_C(A) &= \{w \in W \mid r_C(w) \cap A \neq \emptyset\}. \end{aligned} \tag{9}$$

The set $\underline{Apr}_C(A)$ consists of the elements in W compatible with only those elements in A, while the set $\overline{Apr}_C(A)$ consists of the elements in W compatible with at least one element in A.

The pair of dual mappings given by equation (9) is called an interval structure. An interval structure obey properties similar to (L1)-(L6) and (U1)-(U6). We may axiomatize interval structures by a subset of these properties [56].

Theorem 8 *Suppose $L, H : 2^U \longrightarrow 2^W$ is a pair of dual operators. If H satisfies the following axioms:*

(u1) $H(\emptyset) = \emptyset$,
(u2) $H(A \cup B) = H(A) \cup H(B)$,
(u3) $H(U) = W$,

there exists a relation $C \subseteq W \times U$, with $r_C(w) \neq \emptyset$ for all $w \in W$, such that for all $A \subseteq U$, $L(A) = \underline{Apr}_C(A)$ and $H(A) = \overline{Apr}_C(A)$.

Condition (u3) is required so that the binary relation has the property $r_C(w) \neq \emptyset$ for all $w \in W$. If one removes this constraint on the binary relation, only axioms (u1) and (u2) are needed. This theorem can be considered as a generalization of Theorem 3.

Consider a Pawlak approximation space. We choose W to be $U/\Re$. A compatibility relation between elements of $U/\Re$ and U is defined by: for $E \in U/\Re$ and $x \in U$,

$$E\ C\ x \Longleftrightarrow E = [x]_{\Re}. \tag{10}$$

It intermediately follows that $\underline{Apr}_{\Re}(A) = \underline{Apr}_C(A)$ and $\overline{Apr}_{\Re}(A) = \overline{Apr}_C(A)$ for all $A \subseteq U$. Using equation (1), they can be transformed to $\underline{apr}_{\Re}$ and $\overline{apr}_{\Re}$. Therefore, Pawlak rough set algebra may be interpreted in terms of interval structures. Since interval structures are derived from a binary relation on two universes, they may enlarge the application domain of the theory of rough sets. Recently, Yao *et al.* [63] examined various types of compatibility relations in the study of non-numeric approaches to uncertain reasoning.

4 Set-oriented View

This section presents several set-oriented interpretations based on two distinct definitions of rough sets proposed by Iwinski [15] and Pawlak [38]. We only consider the Pawlak approximation space. The argument can be extended to other types of approximation spaces.

4.1 Pairs of Definable Sets

Iwinski [15] presented an interpretation of rough sets based on a subalgebra of the Boolean algebra 2^U. We choose the subalgebra defined by the set of all definable sets $\text{Def}(apr)$. Given two elements $A_1, A_2 \in \text{Def}(apr)$ with $A_1 \subseteq A_2$, Iwinski called the pair (A_1, A_2) a rough set [15]. In order to distinguish it from other definitions, we call the pair an I-rough set. Let $R(apr)$ be the set of all I-rough sets. Set-theoretic operators on $R(apr)$ can be defined component-wise using standard set operators. For a pair of I-rough sets, we have:

$$(A_1, A_2) \cap (B_1, B_2) = (A_1 \cap B_1, A_2 \cap B_2),$$
$$(A_1, A_2) \cup (B_1, B_2) = (A_1 \cup B_1, A_2 \cup B_2). \tag{11}$$

Such operators are well defined because the intersection and union of two definable sets are definable sets. That is, the results are also I-rough sets. The system $(R(apr), \cap, \cup)$ is a complete distributive lattice [15], with zero element $(\emptyset, \emptyset)$ and unit element (U, U). The associated order relation can be interpreted as I-rough set inclusion, which is defined by:

$$(A_1, A_2) \subseteq (B_1, B_2) \iff A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2. \tag{12}$$

The difference of I-rough sets can be defined as

$$(A_1, A_2) - (B_1, B_2) = (A_1 - B_2, A_2 - B_1), \tag{13}$$

which is an I-rough set. Finally, the I-rough set complement is given as:

$$\sim (A_1, A_2) = (U, U) - (A_1, A_2) = (\sim A_2, \sim A_1). \tag{14}$$

The complement is neither a Boolean complement nor a pseudocomplement in the lattice $(R(apr), \cap, \cup)$. The system $(R(apr), \cap, \cup, \sim, (\emptyset, \emptyset), (U, U))$ is called an I-rough set algebra.

Although such a formulation provides an elegant mathematical model, it is not entirely clear what are the members of an I-rough set. Consequently, set-theoretic operators on $R(apr)$ do not have a well-defined semantics.

4.2 Interval Sets

Given two subsets $A_1, A_2 \subseteq U$ with $A_1 \subseteq A_2$, we define the following closed interval set:

$$[A_1, A_2] = \{X \in 2^U \mid A_1 \subseteq X \subseteq A_2\}, \tag{15}$$

which is a subset of 2^U. The set A_1 is called the lower bound and A_2 the upper bound. That is, members of an interval set are subsets of the universe U. An interval set consists of all those subsets that are bounded by two particular elements of the Boolean algebra 2^U. Let $I(2^U)$ denote the set of all closed interval sets.

Set-theoretic operators on interval sets can be defined based on set operators on their members. For two interval sets $\mathcal{A} = [A_1, A_2]$ and $\mathcal{B} = [B_1, B_2]$, interval set intersection, union, and difference are defined by:

$$\begin{aligned}\mathcal{A} \sqcap \mathcal{B} &= \{X \cap Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\},\\ \mathcal{A} \sqcup \mathcal{B} &= \{X \cup Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\},\\ \mathcal{A} \setminus \mathcal{B} &= \{X - Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}.\end{aligned} \tag{16}$$

The above defined operators are closed on $I(2^U)$, namely, $\mathcal{A} \sqcap \mathcal{B}$, $\mathcal{A} \sqcup \mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ are interval sets. They can be explicitly computed by:

$$\begin{aligned}\mathcal{A} \sqcap \mathcal{B} &= [A_1 \cap B_1, A_2 \cap B_2],\\ \mathcal{A} \sqcup \mathcal{B} &= [A_1 \cup B_1, A_2 \cup B_2],\\ \mathcal{A} \setminus \mathcal{B} &= [A_1 - B_2, A_2 - B_1].\end{aligned} \tag{17}$$

Interval set complement $\neg$ is defined by $[U, U] \setminus [A_1, A_2]$. This is equivalent to $[U - A_2, U - A_1] = [\sim A_2, \sim A_1]$. Clearly, we have $\neg[\emptyset, \emptyset] = [U, U]$ and $\neg[U, U] = [\emptyset, \emptyset]$.

Degenerate interval sets of the form $[A, A]$ are equivalent to ordinary sets. For degenerate interval sets, the proposed operators $\sqcap$, $\sqcup$, $\setminus$, and $\neg$ reduce to set operators. Interval set operators obey most properties of set operators. For example, idempotent, commutativity, associativity, distributivity, and De Morgan's laws hold for $\sqcap$ and $\sqcup$; absorption and double negation laws hold for $\neg$. Thus, the system $(I(2^U), \sqcap, \sqcup)$ is a complete distributive lattice [28], with zero element $[\emptyset, \emptyset]$ and unit element $[U, U]$. The associated order relation is called interval set inclusion. It can be defined using set inclusion relation [60,65]:

$$\mathcal{A} \sqsubseteq \mathcal{B} \iff A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2. \tag{18}$$

The system $(I(2^U), \sqcap, \sqcup, \neg, [\emptyset, \emptyset], [U, U]))$ is called an interval set algebra, which may be considered as a set counterpart of interval number algebra [26,60].

From the above discussion, one can immediately draw a relationship between I-rough set algebra and interval set algebra.

Theorem 9 *Suppose* $R = (R(apr), \cap, \cup, \sim, (\emptyset, \emptyset), (U, U))$ *is an I-rough set algebra, and* $I = (I(2^U), \sqcap, \sqcup, \neg, [\emptyset, \emptyset], [U, U]))$ *is an interval set algebra. The function* $h : R(apr) \longrightarrow I(2^U)$,

$$h((A_1, A_2)) = [A_1, A_2], \tag{19}$$

is a homomorphism from R *to* I.

One may also consider a subalgebra of an interval set algebra in which lower and upper bounds of interval sets must be definable sets. In this case, with respect to the same Pawlak approximation space, an I-rough set algebra uniquely determines an interval set subalgebra, and vice versa. I-rough sets and interval sets may be considered as equivalent algebraically. Consequently, an I-rough set may be viewed as an interval set, which in turn can be viewed as a family of subsets of U. Thus, we associate a well-defined semantics to I-rough set operators. They are extensions of set operators on the members of interval sets [60].

An interval set $[A_1, A_2]$ is also referred to as a flou set [28]. A_1 is called the sure region, A_2 the maximum region, and $A_2 - A_1$ the flou region. The sure and flou regions of an interval set correspond to the positive and boundary regions induced by lower and upper approximations. The notion of interval sets was also used in the study of conditional events [13].

4.3 Families of Subsets

In Pawlak's seminal paper, another set-oriented interpretation of rough sets was introduced. Using lower and upper approximations, we define a binary relation on subsets of U:

$$X \approx Y \Longleftrightarrow \underline{apr}_{\Re}(X) = \underline{apr}_{\Re}(Y) \text{ and } \overline{apr}_{\Re}(X) = \overline{apr}_{\Re}(Y). \tag{20}$$

It is an equivalence relation which induces a partition $2^U/\approx$ of 2^U. An equivalence class of $\approx$ is called a P-rough set. The set of all P-rough sets is denoted by $R_{\approx}(apr) = 2^U/\approx$. More specifically, given two sets $A_1, A_2 \in \text{Def}(apr)$ with $A_1 \subseteq A_2$, a P-rough set is the following family of subsets of U:

$$\langle A_1, A_2\rangle = \{X \in 2^U \mid \underline{apr}_{\Re}(X) = A_1, \overline{apr}_{\Re}(X) = A_2\}. \tag{21}$$

A set $X \in \langle A_1, A_2\rangle$ is said to be a member of the P-rough set. Given a member X, a P-rough set can also be more conveniently expressed as $[X]_{\approx}$, which is the equivalent class containing X. A member is also referred to as a generator of the P-rough set [6].

Rough set intersection $\sqcap$, union $\sqcup$, and complement $\neg$ are defined by set operators as follows: for two P-rough sets $\langle A_1, A_2\rangle$ and $\langle B_1, B_2\rangle$,

$$\begin{aligned}
&\langle A_1, A_2\rangle \sqcap \langle B_1, B_2\rangle \\
&= \{X \in 2^U \mid \underline{apr}_{\Re}(X) = A_1 \cap B_1, \overline{apr}_{\Re}(X) = A_2 \cap B_2\} \\
&= \langle A_1 \cap B_1, A_2 \cap B_2\rangle, \\
&\langle A_1, A_2\rangle \sqcup \langle B_1, B_2\rangle \\
&= \{X \in 2^U \mid \underline{apr}_{\Re}(X) = A_1 \cup B_1, \overline{apr}_{\Re}(X) = A_2 \cup B_2\} \\
&= \langle A_1 \cup B_1, A_2 \cup B_2\rangle,
\end{aligned}$$

$$\begin{aligned}
&\neg\langle A_1, A_2\rangle \\
&= \{X \in 2^U \mid \underline{apr}_{\Re}(X) = \ \sim A_2, \overline{apr}_{\Re}(X) = \ \sim A_1\}, \\
&= \langle \sim A_2, \sim A_1\rangle. \qquad (22)
\end{aligned}$$

The results are also P-rough sets. The induced system $(R_{\approx}(apr), \sqcap, \sqcup)$ is a complete distributive lattice [2,48], with zero element $[\emptyset]_{\approx}$ and unit element $[U]_{\approx}$. The corresponding order relation is called P-rough set inclusion given by:

$$\langle A_1, A_2\rangle \sqsubseteq \langle B_1, B_2\rangle \iff A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2. \qquad (23)$$

The system $(R_{\approx}(apr), \sqcap, \sqcup, \neg, [\emptyset]_{\approx}, [U]_{\approx})$ is called a P-rough set algebra. Follow the same argument, one may define two additional P-rough set algebras by using either the lower or the upper approximation [38]. Each of them defines an equivalence relation on 2^U. By interpreting an equivalence class as a P-rough set, the induced algebras have a similar structure.

Unlike the interval set based interpretation, operations on $R_{\approx}(apr)$ are not defined using all members of P-rough sets. Gehrke and Walker [12] suggested that one may construct a uniform set $\{X_r \mid X \in R_{\approx}(apr)\}$ of representatives, such that

$$\begin{aligned}
&\underline{apr}_{\Re}(X_r \cap Y_r) = \underline{apr}_{\Re}(X_r) \cap \underline{apr}_{\Re}(Y_r), \\
&\underline{apr}_{\Re}(X_r \cup Y_r) = \underline{apr}_{\Re}(X_r) \cup \underline{apr}_{\Re}(Y_r), \\
&\overline{apr}_{\Re}(X_r \cap Y_r) = \overline{apr}_{\Re}(X_r) \cap \overline{apr}_{\Re}(Y_r), \\
&\overline{apr}_{\Re}(X_r \cup Y_r) = \overline{apr}_{\Re}(X_r) \cup \overline{apr}_{\Re}(Y_r).
\end{aligned}$$

For any equivalent class $E \in U/\Re$ with at least two elements, one can select a set S_E such that $\emptyset \subset S_E \subset E$. Let $S = \bigcup_{E \in U/\Re} S_E$. A representative X_r of a P-rough set $\langle A_1, A_2\rangle \in R_{\approx}(apr)$ is defined by $X_r = A_1 \cup (A_2 \cap S)$. Obviously, X_r is a member of $\langle A_1, A_2\rangle$. The triplet $(\{X_r \mid X \in R_{\approx}(apr)\}, \cap, \cup)$ is a sublattice of $(2^U, \cap, \cup)$. It is isomorphic to the lattice formed by all P-rough sets $(R_{\approx}(apr), \sqcap, \sqcup)$. Therefore, P-rough set operators on $R_{\approx}(apr)$ are interpreted by set operators on their representatives. Interpretations of P-rough set operators using the notions of minimum upper samples are essentially alone the same line of argument [1,48].

With the interpretation of P-rough sets as equivalence classes, one may also characterize rough set algebras using other mathematical structures [1,9,12,32,37,48]. For example, a pseudocomplement on $R_{\approx}(apr)$ can be defined as:

$$[A]^*_{\approx} = [U - \overline{apr}_{\Re}(A)]_{\approx}. \qquad (24)$$

The induced system $(R_{\approx}(apr), \sqcap, \sqcup, ^*, [\emptyset]_{\approx}, [U]_{\approx})$ is a complete, atomic Stone algebra [1,48].

4.4 Rough Membership Functions

In the last two subsections, we have discussed interpretations of rough sets in terms of subsets of U. That is, a rough set is viewed as a subset of 2^U. We now turn our attention to interpretations that use elements of U.

4.4.1 A Rough Set as Three Ordinary Sets

A simple and straightforward way for interpreting rough sets is to use three membership functions [61]. A rough set is defined, in terms of elements of U, by a membership function μ_A of the reference set A, and a pair of strong and weak membership functions $\mu_{\underline{apr}_{\Re}(A)}$ and $\mu_{\overline{apr}_{\Re}(A)}$. Let $\mu_{\Re}$ denote the membership function of $\Re$. The strong and weak membership functions of a rough set can be expressed as:

$$\begin{aligned}
\mu_{\underline{apr}_{\Re}(A)}(x) &= \min\{\mu_A(y) \mid y \in [x]_{\Re}\} \\
&= \min\{\max(\mu_A(y), 1 - \mu_{\Re}(x, y)) \mid y \in U\}, \\
\mu_{\overline{apr}_{\Re}(A)}(x) &= \max\{\mu_A(y) \mid y \in [x]_{\Re}\} \\
&= \max\{\min(\mu_A(y), \mu_{\Re}(x, y)) \mid y \in U\}. \qquad (25)
\end{aligned}$$

For two rough sets $(A, \underline{apr}_{\Re}(A), \overline{apr}_{\Re}(A))$ and $(B, \underline{apr}_{\Re}(B), \overline{apr}_{\Re}(B))$, their intersection and union are defined by $(A \cap B, \underline{apr}_{\Re}(A \cap B), \overline{apr}_{\Re}(A \cap B))$ and $(A \cup B, \underline{apr}_{\Re}(A \cup B), \overline{apr}_{\Re}(A \cup B))$, with reference sets $A \cap B$ and $A \cup B$, respectively. The rough set complement is defined by $(\sim A, \underline{apr}_{\Re}(\sim A), \overline{apr}_{\Re}(\sim A))$, with reference set $\sim A$. By definition, rough set intersection and union are not truth-functional. For example, it is impossible to calculate the weak membership function of rough set intersection and the strong membership function of rough set union based merely on the membership functions of two rough sets involved. One must also take into consideration the interaction between two reference sets, and their relationships to equivalent classes of $\Re$. This view can be used to generalize and combine theories of rough and fuzzy sets [10,61].

4.4.2 An I-rough Set as a Fuzzy Set

Consider an I-rough set algebra $(R(apr), \cap, \cup, \neg, (\emptyset, \emptyset), (U, U))$. Using the elements of U, we associate an I-rough set $(A_1, A_2) \in R(apr)$ with the following membership function [40]:

$$\mu_{(A_1,A_2)}(x) = \begin{cases} 1 & x \in A_1, \\ 0.5 & x \in A_2 - A_1, \\ 0 & x \in \sim A_2, \end{cases} \tag{26}$$

With such a membership function, the intersection, union and complement can be expressed component-wise by:

$$\begin{aligned} \mu_{(A_1,A_2)\cap(B_1,B_2)}(x) &= \mu_{(A_1\cap B_1, A_2\cap B_2)}(x) \\ &= \min(\mu_{(A_1,A_2)}(x), \mu_{(B_1,B_2)}(x)), \\ \mu_{(A_1,A_2)\cup(B_1,B_2)}(x) &= \mu_{(A_1\cup B_1, A_2\cup B_2)}(x) \\ &= \max(\mu_{(A_1,A_2)}(x), \mu_{(B_1,B_2)}(x)), \\ \mu_{\sim(A_1,A_2)}(x) &= \mu_{(\sim A_2, \sim A_1)}(x) \\ &= 1 - \mu_{(A_1,A_2)}(x), \end{aligned} \tag{27}$$

where $x \in U$. The membership function $\mu_{(A_1,A_2)}$ may be regarded as defining a fuzzy set. Operators in I-rough set algebra coincide with the standard max-min fuzzy set operators [69]. Let $\mathcal{F}_{0.5}$ denote the set of all functions from U to $\{0, 0.5, 1\}$. Let $\mu_{\mathbf{0}}$ and $\mu_{\mathbf{1}}$ denote, respectively, the functions that uniformly take 0 and 1 for all $x \in U$. We consider a special kind of fuzzy set algebra $(\mathcal{F}_{0.5}, \min, \max, -, \mu_{\mathbf{0}}, \mu_{\mathbf{1}})$, where the operators min, max, and $-$ are defined component-wise similar to equation (27). The following theorem shows that such a system provides a fuzzy set based interpretation of I-rough sets.

Theorem 10 *Suppose* $(R(apr), \cap, \cup, \sim, (\emptyset, \emptyset), (U, U))$ *is an I-rough set algebra, and* $(\mathcal{F}_{0.5}, \min, \max, -, \mu_{\mathbf{0}}, \mu_{\mathbf{1}})$ *a fuzzy set algebra. The function defined component-wise by equation (26) is a homomorphism from set of all I-rough sets* $R(apr)$ *to the set of fuzzy sets* $\mathcal{F}_{0.5}$.

In the interpretation of I-rough sets using a fuzzy sets, every element in the universe can only take one of three possible values. Yao and Li [62] demonstrated that interval set algebra is related to Kleene's three-valued logic. The same connection can also be established between I-rough sets and three-valued logic.

4.4.3 Probabilistic Rough Sets

Pawlak and Skowron [45], Pawlak *et al.* [46], Wong and Ziarko [57], and Yao and Wong [66] proposed another way to characterize a rough set by a single membership function. For any $A \subseteq U$, a rough membership function is defined by:

$$\mu_A(x) = \frac{|A \cap [x]_\Re|}{|[x]_\Re|}, \tag{28}$$

where $|\cdot|$ denotes the cardinality of a set. By definition, elements in the same equivalent class have the same degree of membership. One can see the similarity between rough membership function and conditional probability. The rough membership value $\mu_A(x)$ may be interpreted as the probability of x belonging to A given that x belongs to an equivalence class. Under this interpretation, one obtains the notion of probabilistic rough sets [46,57]. By the law of probability, the intersection and union of probabilistic rough sets are not truth-functional. Nevertheless, we have:

(m1) $\mu_A(x) = 1 \Longleftrightarrow x \in \mathrm{POS}(A)$,
(m2) $\mu_A(x) = 0 \Longleftrightarrow x \in \mathrm{NEG}(A)$,
(m3) $0 < \mu_A(x) < 0 \Longleftrightarrow x \in \mathrm{BND}(A)$,
(m4) $\mu_{\sim A}(x) = 1 - \mu_A(x)$,
(m5) $\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A\cap B}(x)$,
(m6) $\max(0, \mu_A(x) + \mu_B(x) - 1) \leq \mu_{A\cap B} \leq \min(\mu_A(x), \mu_B(x))$,
(m7) $\max(\mu_A(x), \mu_B(x)) \leq \mu_{A\cup B}(x) \leq \min(1, \mu_A(x) + \mu_B(x))$.

They follow from the property of probability.

With the rough membership function (28), one may view a probabilistic rough set as a special type of fuzzy set [57]. By drawing such a link between these two theories, the non-truth-functionality of the operators on probabilistic rough sets may provide more insights into the definition of fuzzy set operators. Since the introduction of the theory of fuzzy sets, the definition of fuzzy set operators has been a controversial issue. Many different proposals have been made, such as the max-min, the probabilistic-like, and the bold intersection and union [66]. In the light of probabilistic rough sets, the membership functions of intersection and union must be computed based on not only the sets involved, but also their interaction with equivalence classes in the approximation space. One may consider the approximation space to be the context that provides semantics of the operators on probabilistic rough sets. From this observation, we may say that a plausible definition of fuzzy set operators must also incorporate semantics information about the fuzzy concepts being modeled. The probabilistic-rough set model provides only one of many solutions. In general, it seems unreasonable to assume that there is a universal way of defining fuzzy set operators. The correct choice of fuzzy set operators depends on nature of the physical problem one attempts to model. It may be fruitful to study fuzzy set models with non-truth-functional operators.

The notion of probabilistic rough sets may be related to P-rough set algebra $(R_{\approx}(apr), \sqcap, \sqcup, \neg, [\emptyset]_{\approx}, [U]_{\approx})$. For two members of the same P-rough set, i.e., $A \approx B$, they may not be characterized by the same membership function, i.e., $\mu_A \neq \mu_B$. Let $c(\mu_A)$ and $s(\mu_A)$ denote the core and support of μ_A defined by:

$$\begin{aligned} c(\mu_A) &= \{x \mid \mu_A(x) = 1\}, \\ s(\mu_A) &= \{x \mid \mu_A(x) > 0\}. \end{aligned} \tag{29}$$

By properties (m1) and (m2), one can verify that if $A \approx B$, then $c(\mu_A) = c(\mu_B)$ and $s(\mu_A) = s(\mu_B)$. In other words, a P-rough set is a family of probabilistic rough sets with the same core and support.

5 Conclusion

The successful application of the theory of rough sets depends on a clear understanding of the various concepts involved. There at least two views that can be adopted for interpreting the theory of rough sets. Both of these views can be explained using the notion of lower and upper approximations in an approximations space. They differ from the way in which these approximations are used. If approximations are adopted to construct operators, the operator-oriented view is obtained. The theory of rough sets is therefore an extension of set theory with two additional unary operators. Alternatively, the lower and upper approximations are used to define the notion of rough sets. A rough set can be defined by using either subsets of the universe or elements of subsets. Under this interpretation, no additional set-theoretic operators are introduced. Instead, the standard set-theoretic operators are modified to capture the new semantics required by the theory of rough sets.

Each of the proposed views captures different and important aspects of the notion of rough sets. Using the proposed views, we have investigated the connections between the theory of rough sets and other theories of uncertainty. The operator-oriented view is related to topological space, modal logics, Boolean algebra with added operators, and interval structures. The set-oriented view is related to interval sets, conditional events, and fuzzy sets. Both views are useful in many sub-area of artificial intelligence. The operator-oriented view is more suitable for approximate reasoning, while the set-oriented view is more convenient in classification, concept formation, and information system analysis. The results of our investigation show that the theory of rough sets is very rich and versatile. The established connections to many other theories may lead to new applications of theory of rough sets.

In this paper, approximation spaces are formulated in the context of set theory. The argument can be easily applied to develop more generalized theories of rough sets using other mathematical structures, such as Boolean algebras and lattices [8,12,56].

References

[1] Bonikowski, Z., A certain conception of the calculus of rough sets, *Notre Dame J. Form. Log.* 33, 412-421, 1992.

[2] Bonikowski, Z., Algebraic structures of rough sets, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 242-247, 1994.

[3] Bryniarski, E., A calculus of rough sets of the first order, *Bull. Pol. Acad. Sci. Math.* 37, 71-77, 1989.

[4] Bundy, A., Incidence calculus: a mechanism for probabilistic reasoning, *J. Autom. Reasoning* 1, 263-283, 1985.

[5] Chakraborty, M.K., and Banerjee, M., Rough consequence and rough algebra, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 196-207, 1994.

[6] Chanas, S., and Kuchta, D., Further remarks on the relation between rough and fuzzy sets, *Fuzzy Sets Syst.* 47, 391-394, 1992.

[7] Chellas, B.F., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

[8] Chuchro, M., On rough sets in topological Boolean algebras, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 157-166, 1994.

[9] Comer, S., On connections between information systems, rough sets, and algebraic logic, in *Algebraic Methods in Logic and Computer Science* (C. Rauszer, Ed.), Banach Center Publications, Vol. 28, Polish Academy of Sciences, 117-124, 1993.

[10] Dubois, D., and Prade, H., Rough fuzzy sets and fuzzy rough sets, *Int. J. Gen. Syst.* 17, 191-209, 1990.

[11] Fagin, R., and Halpern, J.Y., Uncertainty, belief, and probability, *Comput. Intell.* 7, 160-173, 1991.

[12] Gehrke, M., and Walker, E., On the structure of rough sets, *Bull. Pol. Acad. Sci. Math.* 40, 235-245, 1992.

[13] Goodman, I., Nguyen, H., and Walker, E., *Conditional Inference and Logic for Intelligent Systems*, North-Holland, New York, 1991.

[14] Hughes, G.E., and Cresswell, M.J., *An Introduction to Modal Logic*, Methuen, London, 1968.

[15] Iwinski, T.B., Algebraic approach to rough sets, *Bull. Pol. Acad. Sci. Math.* 35, 673-683, 1987.

[16] Kortelainen, J., On relationship between modified sets, topological spaces and rough sets, *Fuzzy Sets Syst.* 61, 91-95, 1994.

[17] Kramosil, I., Definition and recognition of classical sets by the rough ones, *Problems of Control and Information Theory* 20, 77-95, 1991.

[18] Lemmon, E.J., Algebraic semantics for modal logics I, II, *J. Symb. Log.* 31, 46-65, 191-218, 1966.

[19] Lin, T.Y., Topological and fuzzy rough sets, in *Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory* (R. Slowinski, Ed.), Kluwer Academic Publishers, Boston, 287-304, 1992.

[20] Lin, T.Y., Fuzzy reasoning and rough sets, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 343-348, 1994.

[21] Lin, T.Y. (Ed.), *Proceedings of the CSC'95 Workshop on Rough Sets and Database Mining*, San Jose State University, 1995.

[22] Lin, T.Y., and Liu, Q., Rough approximate operators: axiomatic rough set theory, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 256-260, 1994.

[23] Lin, T.Y., and Liu, Q., First order rough logic I: approximate reasoning via rough sets, *Fundam. Inform.*, to appear.

[24] Lin, T.Y., Liu, Q., and Huang, K.J., and Chen, W., Rough sets, neighborhood systems and approximation, *Proceedings of the Fifth International Symposium on Methodologies of Intelligent Systems*, 130-141, 1990.

[25] Lin, T.Y., and Wildberger, A.M. (Eds.), *Soft Computing*, The Society for Computer Simulation, San Diego, 1995.

[26] Moore, R.E., *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1966.

[27] Nakamura, A., and Gao, J.M., A logic for fuzzy data analysis, *Fuzzy Sets Syst.* 39, 127-132, 1991.

[28] Negoiţă, C.V., and Ralescu, D.A., *Applications of Fuzzy Sets to Systems Analysis*, Birkhäuser Verlag, Basel, 1975.

[29] Nguyen, H.T., Intervals in Boolean rings: approximation and logic, *Found. Comput. Decis. Sci.* 17, 131-138, 1992.

[30] Nguyen, H.T., and Walker, E.A., A history and introduction to the algebra of conditional events and probability logic, *IEEE Trans. Syst. Man Cybern.* 24, 1671-1675, 1994.

[31] Novotny, M., and Pawlak, Z., On rough equalities, *Bull. Pol. Acad. Sci. Math.* 33, 99-113, 1985.

[32] Obtulowicz, A., Rough sets and Heyting algebra valued sets, *Bull. Pol. Acad. Sci. Math.* 35, 667-671, 1987.

[33] Orlowska, E., Logic of indiscernibility relations, *Bull. Pol. Acad. Sci. Math.* 33, 475-485, 1985.

[34] Orlowska, E., Logical aspects of learning concepts, *Int. J. Approx. Reason.* 2, 349-364, 1988.

[35] Orlowska, E., Rough set semantics for non-classical logics, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 143-148, 1994.

[36] Orlowska, E., and Pawlak, Z., Measurement and indiscernibility, *Bull. Pol. Acad. Sci. Math.* 32, 617-624, 1984.

[37] Pagliani, P., A pure logic-algebraic analysis of rough top and rough bottom equalities, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 225-241, 1994.

[38] Pawlak, Z., Rough sets, *Int. J. Comp. & Inf. Sci.* 11, 341-356, 1982.

[39] Pawlak, Z., Rough classification, *Int. J. Man-Mach. Stud.* 20, 469-483, 1984.

[40] Pawlak, Z., Rough sets and fuzzy sets, *Fuzzy Sets Syst.* 17, 99-102, 1985.

[41] Pawlak, Z., Rough logic, *Bull. Pol. Acad. Sci. Tech. Sci.* 35, 253-258, 1987.

[42] Pawlak, Z., *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, Boston, 1991.

[43] Pawlak, Z., Rough sets: a new approach to vagueness, in *Fuzzy Logic for the Management of Uncertainty* (L.A. Zadeh and J. Kacprzyk, Eds.), John Wiley & Sons, New York, 105-118, 1992.

[44] Pawlak, Z., Hard and soft sets, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 130-135, 1994.

[45] Pawlak, Z., and Skowron, A., Rough membership functions, in *Fuzzy Logic for the Management of Uncertainty* (L.A. Zadeh and J. Kacprzyk, Eds.), John Wiley & Sons, New York, 251-271, 1994.

[46] Pawlak, Z., Wong, S.K.M., and Ziarko, W., Rough sets: probabilistic versus deterministic approach, *Int. J. Man-mach. Stud.* 29, 81-95, 1988.

[47] Pomykala, J.A., Approximation operations in approximation space, *Bull. Pol. Acad. Sci. Math.* 35, 653-662, 1987.

[48] Pomykala, J., and Pomykala, J.A., The Stone algebra of rough sets, *Bull. Pol. Acad. Sci. Math.* 36, 495-508, 1988.

[49] Rasiowa, H., *An Algebraic Approach to Non-classical Logics*, North-Holland, Amsterdam, 1974.

[50] Shafer, G., Belief functions and possibility measures, in *Analysis of Fuzzy Information, vol. 1: Mathematics and Logic* (J.C. Bezdek, Ed.), CRC Press, Boca Raton, 51-84, 1987.

[51] Skowron, A., The rough sets theory and evidence theory, *Fundam. Informat.* XIII, 245-262, 1990.

[52] Skowron, A., and Grzymala-Busse, J., From rough set theory to evidence theory, in *Advances in the Dempster-Shafer Theory of Evidence*, (R.R. Yager, M. Fedrizzi and J. Kacprzyk, Eds.), John Wiley & Sons, New York, 193-236, 1994.

[53] Slowinski, R. (Ed.), *Intelligent Decision Support: Handbook of Applications and Advances of the Rough Sets Theory*, Kluwer Academic Publishers, Boston, 1992.

[54] Wasilewska, A., Topological rough algebras, Manuscript, 1995.

[55] Wiweger, A., On topological rough sets, *Bull. Pol. Acad. Sci. Math.* 37, 89-93, 1989.

[56] Wong, S.K.M., Wang, L.S., and Yao, Y.Y., On modeling uncertainty with interval structures, *Comput. Intell.* 11, 406-426, 1995.

[57] Wong, S.K.M., and Ziarko, W., Comparison of the probabilistic approximate classification and the fuzzy set model, *Fuzzy Sets Syst.* 21, 357-362, 1987.

[58] Wybraniec-Skardowska, U., On a generalization of approximation space, *Bull. Pol. Acad. Sci. Math.* 37, 51-61, 1989.

[59] Wygralak, M., Rough sets and fuzzy sets – some remarks on interrelations, *Fuzzy Sets Syst.* 29, 241-243, 1989.

[60] Yao, Y.Y., Interval-set algebra for qualitative knowledge representation, *Proceedings of the Fifth International Conference on Computing and Information*, 370-375, 1993.

[61] Yao, Y.Y., On combining rough and fuzzy sets, *Proceedings of the CSC'95 Workshop on Rough Sets and Database Mining*, Lin, T.Y. (Ed.), San Jose State University, 9 pages, 1995.

[62] Yao, Y.Y., and Li, X., Uncertain reasoning with interval-set algebra, in *Rough Sets, Fuzzy Sets and Knowledge Discovery* (W.P. Ziarko, Ed.), Springer-Verlag, London, 178-185, 1994.

[63] Yao, Y.Y., Li, X., Lin, T.Y., and Liu, Q., Representation and classification of rough set models, in *Soft Computing* (T.Y. Lin and A.M. Wildberger, Eds.), The Society for Computer Simulation, San Diego, 44-47, 1995.

[64] Yao, Y.Y., and Lin, T.Y., Generalization of rough sets using modal logic, *Intell. Autom. and Soft Comput. Int. J.*, to appear.

[65] Yao, Y.Y., and Noroozi, N., A unified model of set-based computation, in *Soft Computing* (T.Y. Lin, and A.M. Wildberger, Eds.), The Society for Computer Simulation, San Diego, 252-255, 1995.

[66] Yao, Y.Y., and Wong, S.K.M., A decision theoretic framework for approximating concepts, *Int. J. Man-mach. Stud.* 37, 793-809, 1992.

[67] Yao, Y.Y., Wong, S.K.M., and Wang, L.S., A non-numeric approach to uncertain reasoning, *Int.. J. Gen. Syst.* 23, 343-359, 1995.

[68] Zakowski, W., Approximations in the space (U, Π), *Demonstratio Math.* XVI, 761-769, 1983.

[69] Zadeh, L.A., Fuzzy sets, *Inf. & Contr.* 8, 338-353, 1965.

[70] Ziarko, W.P. (Ed.), *Rough Sets, Fuzzy Sets and Knowledge Discovery*, Springer-Verlag, London, 1994.

Appendix: Proofs of Theorems

Proof of Theorem 3.

Suppose operator H obeys axioms (c1) and (c2). Using H, we can construct a binary relation as follows:

$$x\Re y \Longleftrightarrow x \in H(\{y\}).$$

That is, $r(x) = \{y \mid x \in H(\{y\})\}$, and conversely $H(\{y\}) = \{x \mid y \in r(x)\}$. By definition and property (c1), for the empty set we have $\overline{apr}_{\Re}(\emptyset) = \emptyset = H(\emptyset)$. For singleton subsets, it follows:

$$\begin{aligned}\overline{apr}_{\Re}(\{y\}) &= \{x \mid r(x) \cap \{y\} \neq \emptyset\} \\ &= \{x \mid y \in r(x)\} \\ &= H(\{y\}).\end{aligned}$$

By axiom (c2) and the fact that U is finite, we have:

$$\begin{aligned}\overline{apr}_{\Re}(A) &= \bigcup_{y \in A} \overline{apr}_{\Re}(\{y\}) \\ &= \bigcup_{y \in A} H(\{y\}) \\ &= H(A),\end{aligned}$$

for all subsets $A \subseteq U$. □

Proof of Theorem 5.

In order to prove this theorem, we first state the following results. Suppose $\Re$ is a binary relation on a set U. Then: for all $x, y \in U$,

$$\begin{aligned}
\Re \text{ is reflexive } &\Longleftrightarrow x \in r(x), \\
\Re \text{ is symmetric } &\Longleftrightarrow [x \in r(y) \Longrightarrow y \in r(x)], \\
\Re \text{ is transitive } &\Longleftrightarrow [y \in r(x) \Longrightarrow r(y) \subseteq r(x)].
\end{aligned}$$

According to the proof of Theorem 3, axioms (c1) and (c2) guarantee the existence of a binary relation $\Re$ such that $\overline{apr}_{\Re}(A) = H(A)$ for all $A \subseteq U$. Now we show that $\Re$ is an equivalence relation. For any element $x \in U$, by axiom (c3), we have $\{x\} \subseteq H(\{x\})$. That is, $x \in H(\{x\})$. From $H(\{x\}) = \{y \mid x \in r(y)\}$, we have $x \in r(x)$. Thus, $\Re$ is reflexive. Assume $x \in r(y)$ for two elements $x, y \in U$. By axiom (c5), we have $\{y\} \subseteq \sim H(\sim H(\{y\}))$. It follows that $y \in \sim H(\sim H(\{y\}))$, which is equivalent to $y \notin H(\sim H(\{y\}))$. Thus, $r(y) \cap \sim H(\{y\}) = \emptyset$, namely, $r(y) \subseteq H(\{y\})$. From the assumption $x \in r(y)$, we have $x \in H(\{y\})$. By definition, $y \in r(x)$. Therefore, one can conclude that $\Re$ is symmetric. Assume $y \in r(x)$ for two elements $x, y \in U$. Assume further $z \in r(y)$. By definition, $y \in H(\{z\})$, which implies $\{y\} \subseteq H(\{z\})$. Axiom (c2) implies that the operator H is monotonic with respect to set inclusion. Thus, $H(\{y\}) \subseteq H(H(\{z\}))$. From axiom (c4), we have $H(\{y\}) \subseteq H(\{z\})$. The assumption $y \in r(x)$ implies $x \in H(\{y\})$. Hence, $x \in H(\{z\})$, i.e., $z \in r(x)$. Therefore, $r(y) \subseteq r(x)$. Therefore, $\Re$ is transitive. $\Box$