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Interval Sets and Interval-Set Algebras

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Abstract

An interval set is an interval in the power set lattice based on a universal set and is a family of subsets of the universal set. Interval sets and interval-set algebras provide a tool for modeling and processing partially known concepts and for approximating undefinable or complex concepts. Existing results on interval sets and interval-set algebras are reviewed and new results are given. Two types of interval-set algebras are examined based on an inclusion ordering and a knowledge ordering, respectively. Related studies are summarized.

1. Introduction

A concept, in the classical view, is defined by a pair of intension and extension [34]. The extension consists of the instances to which the concept applies; the intension is a set of singly necessary and jointly sufficient conditions that describe the instances of the concept. Concepts are assumed to have well-defined boundaries and their extensions can be precisely defined by sets of objects. Based on the intension-exetension interpretation, in this paper we sometimes use the terms concepts and sets (i.e., extensions of concepts) interchangeably.

Concepts, in many practical situations, cannot be precisely defined and their extensions may not be given exactly by sets of objects. For this reasons, many proposals have been made to generalize the notion of sets for representing inexact, imprecise, vague, or partially known concepts. In generalizing standard, crisp sets for representing real-world concepts, one may consider at least the following scenarios:

- **Concepts with grey boundaries**. There is not a well-defined boundary that differentiates the instances from the non-instances of the concept. For some objects, the concept only applies partially instead of fully. There may exist a sequence of objects that gradually change from non-instances to instances of the concept. Typically, one can define such degree of belongingness in quantitative terms.
- **Partially known concepts**. In some situations, an object may actually be either an instance or not an instance of a concept. On the other hand, due to a lack of information and knowledge, one can only express the state of instance and non-instance for some objects, instead of all objects. That is, one has a partially known concept defined by a lower bound and upper bound of its extension.
- **Undefinable concepts and approximations**. In general, the intension of a concept may be defined by using a logical language, such as the decision logic language in rough set theory [23] and knowledge representation languages of description logic [1]. It may happen that some objects cannot be differentiated due to the use of a fixed and limited set of attributes for their description. The language may not be able to define certain sets of objects that are the extensions of some concepts. That is, we have some undefinable concepts with respect to a particular language. We have to approximate undefinable concepts by definable concepts [15, 23, 47].
- **Complex concepts and approximations**. In this case, one knows the exact extension of a concept and the language can precisely define the concept. However, the description may be too complex to be of any practical value; it may be difficult to understand and manipulate. This may require us to approximate the concept by some concepts with simple descriptions.

Each of these scenarios represents a different semantics interpretation of inexact, imprecise or vague concepts.

These non-classical semantics of concepts require distinct generalizations of standard sets. The theory of fuzzy sets provides a generalization of crisp sets for representing concepts with grey boundaries [54]. The last three scenarios are suggested and discussed by Marek and Truszczyński [15] for motivating the theory of rough sets. A closer examination of Pawlak's proposal [22, 23] shows that the theory of rough sets was originally formulated as a model for approximating undefinable concepts by definable concepts. To some extent, it also covers approximation of complex concepts by simple concepts. Studies of undefinable concepts, complex concepts and approximations are closely related. Practically speaking, concepts with very complex descriptions may be viewed as undefinable concepts and concepts with simple descriptions as definable concepts. Shadowed sets [25], as approximations of fuzzy sets, provide another appropriate model for studying approximation of concepts with grey boundaries.

Partially known concepts may be considered as qualitative grey concepts, which can be modeled by three-valued logic [50]. The boundary of a partially known concept is made up objects whose belongingness to the concept are unknown. Interval sets seem to provide a more suitable model for studying partially known concepts [42, 50, 52, 53].

The main objective of this paper is to review main results and to report new results on interval sets and interval-set algebras. In Section 2, two semantics interpretations of interval sets are examined. Interval sets can be used for modeling and processing partially known concepts, and for approximating undefinable or complex concepts. In Section 3, operations on interval sets are defined based on the notion of lifted operations in power algebras [2]. In Section 4, two kinds of interval-set algebras are investigated based on an inclusion ordering and a knowledge ordering [15, 36], respectively. Section 5 reviews additional studies related to interval sets.

2. Interval Sets

Interval sets are defined and interpreted in a similar way that interval numbers are introduced in interval analysis [18]. Interval sets and associated algebras may be considered as counterparts of interval numbers and interval arithmetics. The results from the latter can be used in the study of the former [42].

Interval sets and interval-set algebras are similar to flou sets and systems [8, 20] in terms of their mathematical structures. However, our emphasis is on the semantics interpretations. A flou set is treated as a pair of sets or a three-valued membership function [8, 20]. Although an interval set corresponds to a pair of sets, it consists of a family of sets. Interval sets are constructed based on two specific semantics interpretations.

2.1. Interval sets for modeling partially known concepts

The notion of interval sets is a new kind of sets, represented by a pair of sets, namely, its lower and upper bounds [42, 50, 52, 53]. Mathematically, interval sets are defined as follows. Let U be a finite set, called the universe or the reference set, and 2^U be its power set. A subset of 2^U of the form,

$$\mathscr{A} = [A_l, A_u] = \{A \in 2^U \mid A_l \subseteq A \subseteq A_u\}, \tag{1}$$

is called a closed interval set, where it is assumed that $A_l \subseteq A_u$. Being an interval of the power set lattice 2^U, an interval set $\mathscr{A}$ is also a lattice, with the minimum element A_l, the maximum element A_u, and the standard set-theoretic operations. The set of all closed interval sets is denoted by:

$$I(2^U) = \{[A_l, A_u] \mid A_l, A_u \subseteq U, A_1 \subseteq A_u\}. \tag{2}$$

Degenerate interval sets of the form $[A,A]$ are equivalent to ordinary sets.

Semantically, an interval set, when interpreted as a family of sets of objects, provides an appropriate means to represent a partially known concept [41, 42]. Although the extension of a concept is actually a subset of U, a lack of knowledge makes us unable to specify this subset. We can only provide a lower bound A_l and an upper bound A_u. Any subset A that lies between A_l and A_u, namely, $A_l \subseteq A \subseteq A_u$, can be the actual extension of the concept. The set,

$$\mathrm{BND}([A_l, A_u]) = A_u - A_l, \tag{3}$$

is called the boundary of the interval set $[A_l, A_u]$. For those elements, we are unable to tell if they are instances or non-instances of the concept.

Interval sets are subsets of the universe U. The symbols $\in, \subseteq, =, \cap, \cup$ may be used, in their usual set-theoretic sense, to represent relationships between elements of 2^U and an interval set, and between different interval sets. Thus, $A \in [A_l, A_u]$ means that A is a subset of U such that $A_l \subseteq A \subseteq A_u$. We write $[A_l, A_u] \subseteq [B_l, B_u]$ if the interval set $[A_l, A_u]$ as an ordinary set is contained in $[B_l, B_u]$ as an ordinary set. In other words, by $[A_l, A_u] \subseteq [B_l, B_u]$ we mean that $B_l \subseteq A_l \subseteq A_u \subseteq B_u$. Similarly, two interval sets are equal, written $\mathscr{A} = \mathscr{B}$, if they are equal in set-theoretic sense, that is $\mathscr{A} = \mathscr{B}$ if and only if $A_l = B_l$ and $A_u = B_u$.

Consider an example for interpreting interval set. Suppose U is a set of patients and $A \subseteq U$ is the subset of patients having a particular disease. Typically, it may be impossible to identify the set A based on observable symptoms, available tests results, and current medical knowledge. The set A_l consists of patients who for sure have the disease; the set $U - A_u$ consists of patients who do not have the disease; the set $A_u - A_l$ consists patients who may or may not have the disease but we are unable to tell which is the case. We only have a pair of bounds for A and any subset that lies within the bounds may be the actual set A.

Consider now another example. Let U be the set of papers submitted to a conference. After the first round of review, one may form three sets, the set of accepted papers A_l, the set of rejected papers $U - A_u$, and the set of papers subject to a second round of review $A_u - A_l$. Although papers in $A_u - A_l$ will eventually be either accepted or rejected, one has no knowledge about it at the current stage. One has to make plan based on an interval set $[A_l, A_u]$ at this stage, with an assumption that any subset A that lies within A_l and A_u may be the set of finally accepted papers.

2.2. Interval sets for approximating undefinable or complex concepts

The discussions of interval sets so far have not made any assumptions on the two ending point of an interval set. This enables us to consider the family of all possible interval sets. When using interval sets to approximate undefinable or complex concepts, we only consider families of interval sets whose ending sets satisfy certain conditions. For modeling undefinable concepts, the ending sets of an interval set must be extensions of definable concepts; for modeling approximate concepts, the ending sets must be extensions of simple concepts. In the context of rough set theory, the former was suggested by Iwiński [7] and studied by Yao [42, 44], and the latter was suggested by Marek and Truszczyński [15]. The former interpretation is also consistent with the interpretation of interval numbers; the two ending numbers of an interval number are numbers that must be representable in a computer.

In rough set theory, a family of definable sets is constructed from an equivalence relation (i.e., a reflexive, symmetric, and transitive relation). Consider an equivalence relation E on the universe U. It induces a partition of the universe and is denoted by U/E. From U/E, we can construct an σ-algebra, $\sigma(U/E)$, which contains the empty set $\emptyset$, equivalence classes of E, and is closed under set intersection, union and complement. The partition U/E is a base of $\sigma(U/E)$. The σ-algebra $\sigma(U/E)$ consists of all definable subsets of U. Detailed discussions on the construction of equivalence relations in an information table and the definability of subsets in $\sigma(U/E)$ can be found in references [15, 23, 47].

For a pair of definable sets $A_l, A_u \in \sigma(U/E)$ with $A_l \subseteq A_u$, an interval set is defined by:

$$\begin{aligned} \mathscr{A} &= [A_l, A_u] \\ &= \{A \in 2^U \mid A_l \subseteq A \subseteq A_u, \\ &\qquad A_l, A_u \in \sigma(U/E)\}. \end{aligned} \tag{4}$$

This interval set corresponds to the pair of definable sets (A_1, A_u) that was called a rough set first by Iwiński [7] in a sense different from Pawlak's rough sets. Yao [44] referred to the pair (A_l, A_u) as an Iwiński rough set to be differentiated from a Pawlak rough set. Marek and Truszczyński [15] called the pair $\langle A_l, A_u \rangle$ an approximation of any subset of $A \subseteq U$ such that $A_l \subseteq A \subseteq A_u$ or a rough set. Furthermore, they showed that every Iwiński rough set is a Pawlak rough set and vice versa if and only if every equivalence class contains at least two objects.

With respect to the family of definable sets $\sigma(U/E)$, the set of all closed interval sets is denoted by:

$$IR(2^U) = \{[A_l, A_u] \mid A_l, A_u \in \sigma(U/E), A_1 \subseteq A_u\}. \tag{5}$$

In general, we have $I(2^U) \subset IR(2^U)$. An interval set in $I(2^U)$ represents a partially known concept. An interval set in $IR(2^U)$ represents an approximation of an undefinable set. There are subtle semantics differences between those interpretations of interval sets.

Alternatively, one may construct an σ-algebra in which each subset representing a concept with a simple description. A subset not in the σ-algebra represent a concept with a complex description. The same argument can be easily applied to this case where we can approximate a complex concept by a pair of simple concepts.

3. Interval Set Operations

Recall that an interval set is interpreted as a set of sets. Interval-set operations can be defined as lifted operations in a power algebraic setting [42].

3.1. Power algebras

The notion of power algebras was proposed and studied by Brink [2]. Let U be a set and $\circ$ a binary operation on U. One can define a binary operation $\circ^+$ on subsets of U as follows [2]:

$$X \circ^+ Y = \{x \circ y \mid x \in X, y \in Y\}, \tag{6}$$

for any $X, Y \subseteq U$. In general, one may lift any operation f on elements of U to an operation f^+ on subsets of U, called the power operation of f. Suppose $f : U^n \longrightarrow U$

($n \geq 1$) is an n-ary operation on U. The power operation $f^+ : (2^U)^n \longrightarrow 2^U$ is defined by [2]:

$$f^+(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n) \mid x_i \in X_i \text{ for } i = 1, \ldots, n\}, \quad (7)$$

for any $X_1, \ldots, X_n \subseteq U$. This provides universal-algebraic construction approach. For any algebra $(U, f_1, \ldots, f_k)$ with base set U and operations $f_1, \ldots, f_k$, its power algebra is given by $(2^U, f_1^+, \ldots, f_k^+)$.

The power operation f^+ may carry some properties of f. For example, for a binary operation $f : U^2 \longrightarrow U$, if f is commutative and associative, f^+ is commutative and associative, respectively. If e is an identity for some operation f, the set $\{e\}$ is an identity for f^+. If an unary operation $f : U \longrightarrow U$ is an involution, i.e., $f(f(x)) = f(x)$, f^+ is also an involution. On the other hand, many properties of f are not carried over by f^+. For instance, if a binary operation f is idempotent, i.e., $f(x,x) = x$, f^+ may not be idempotent. If a binary operation g is distributive over f, g^+ may not be distributive over f^+.

A special type of power algebra is called interval algebra, in which operations on elements of a poset U are lifted to intervals of U, instead of arbitrary subsets of U. In doing so, the power operation f^+ may carry additional properties of f. The notion of interval algebras forms a basis of uncertain reasoning with intervals. Interval-set algebras may be considered to be a special case.

3.2. Interval-set operations as lifted operations

Let $\cap, \cup$ and $-$ be the usual set intersection, union and difference defined on 2^U, respectively. Following the results of power algebras, we can lift set operations into interval-set operations. Specifically, for two interval sets $\mathscr{A} = [A_l, A_u]$ and $\mathscr{B} = [B_l, B_u]$ we have:

$$\begin{aligned} \mathscr{A} \sqcap \mathscr{B} &= \{A \cap B \mid A \in \mathscr{A}, B \in \mathscr{B}\}, \\ \mathscr{A} \sqcup \mathscr{B} &= \{A \cup B \mid A \in \mathscr{A}, B \in \mathscr{B}\}, \\ \mathscr{A} \setminus \mathscr{B} &= \{A - B \mid A \in \mathscr{A}, B \in \mathscr{B}\}. \end{aligned} \quad (8)$$

These operations are referred to as interval-set intersection, union and difference. They are closed on $I(2^U)$, namely, $\mathscr{A} \sqcap \mathscr{B}$, $\mathscr{A} \sqcup \mathscr{B}$ and $\mathscr{A} \setminus \mathscr{B}$ are interval sets. In fact, these interval sets can be explicitly computed by using the following formulas:

$$\begin{aligned} \mathscr{A} \sqcap \mathscr{B} &= [A_l \cap B_l, A_u \cap B_u], \\ \mathscr{A} \sqcup \mathscr{B} &= [A_l \cup B_l, A_u \cup B_u], \\ \mathscr{A} \setminus \mathscr{B} &= [A_l - B_u, A_u - B_l]. \end{aligned} \quad (9)$$

Similarly, the interval-set complement $\neg[A_l, A_u]$ of $[A_l, A_u]$ is defined as $[U, U] \setminus [A_l, A_u]$. This is equivalent to $[U - A_u, U - A_l] = [A_u^c, A_l^c]$, where $A^c = U - A$ denote the usual set complement operation. Clearly, we have $\neg[\emptyset, \emptyset] = [U, U]$ and $\neg[U, U] = [\emptyset, \emptyset]$.

Based on these operations, one can show the correspondence between systems of interval sets, Iwiński rough sets and flou sets. All of them share similar mathematical structures, but with different semantics interpretations.

For operations $\sqcap, \sqcup$ and $\neg$, the following properties hold: for $\mathscr{A}, \mathscr{B}, \mathscr{C} \in I(2^U)$,

(I1) Idempotent :
$\mathscr{A} \sqcap \mathscr{A} = \mathscr{A}$,
$\mathscr{A} \sqcup \mathscr{A} = \mathscr{A}$;

(I2) Commutativity :
$\mathscr{A} \sqcap \mathscr{B} = \mathscr{B} \sqcap \mathscr{A}$,
$\mathscr{A} \sqcup \mathscr{B} = \mathscr{B} \sqcup \mathscr{A}$;

(I3) Associativity :
$(\mathscr{A} \sqcap \mathscr{B}) \sqcap \mathscr{C} = \mathscr{A} \sqcap (\mathscr{B} \sqcap \mathscr{C})$,
$(\mathscr{A} \sqcup \mathscr{B}) \sqcup \mathscr{C} = \mathscr{A} \sqcup (\mathscr{B} \sqcup \mathscr{C})$,

(I4) Distributivity :
$\mathscr{A} \sqcap (\mathscr{B} \sqcup \mathscr{C}) = (\mathscr{A} \sqcap \mathscr{B}) \sqcup (\mathscr{A} \sqcap \mathscr{C})$,
$\mathscr{A} \sqcup (\mathscr{B} \sqcap \mathscr{C}) = (\mathscr{A} \sqcup \mathscr{B}) \sqcap (\mathscr{A} \sqcup \mathscr{C})$;

(I5) Absorption :
$\mathscr{A} \sqcap (\mathscr{A} \sqcup \mathscr{B}) = \mathscr{A}$,
$\mathscr{A} \sqcup (\mathscr{A} \sqcap \mathscr{B}) = \mathscr{A}$;

(I6) De Morgan's laws :
$\neg(\mathscr{A} \sqcap \mathscr{B}) = \neg\mathscr{A} \sqcup \neg\mathscr{B}$,
$\neg(\mathscr{A} \sqcup \mathscr{B}) = \neg\mathscr{A} \sqcap \neg\mathscr{B}$;

(I7) Double negation law :
$\neg\neg\mathscr{A} = \mathscr{A}$,

(I8) $[U, U]$ and $[\emptyset, \emptyset]$ are the unique identities for interval-set intersection and union, that is,
$\mathscr{A} = \mathscr{X} \sqcap \mathscr{A} = \mathscr{A} \sqcap \mathscr{X}$ for all $\mathscr{A} \in I(2^U)$
$\Longleftrightarrow \mathscr{X} = [U, U]$,
$\mathscr{A} = \mathscr{Y} \sqcup \mathscr{A} = \mathscr{A} \sqcup \mathscr{Y}$ for all $\mathscr{A} \in I(2^U)$
$\Longleftrightarrow \mathscr{Y} = [\emptyset, \emptyset]$.

These properties may be regarded as the counterparts of the properties of the corresponding set-theoretic operations. However, for an interval set $\mathscr{A}$, $\mathscr{A} \sqcap \neg\mathscr{A}$ is not necessarily equal to $[\emptyset, \emptyset]$, $\mathscr{A} \sqcup \neg\mathscr{A}$ is not necessarily equal to $[U, U]$, and $\mathscr{A} \setminus \mathscr{A}$ is not necessarily equal to $[\emptyset, \emptyset]$. Nevertheless, the following properties hold:

(I9) $\emptyset \in \mathscr{A} \sqcap \neg\mathscr{A}$,
$U \in \mathscr{A} \sqcup \neg\mathscr{A}$,
$\emptyset \in \mathscr{A} \setminus \mathscr{A}$.

Therefore, $I(2^U)$ is a completely distributive lattice but not a Boolean algebra [20].

The proposed operations $\sqcap$, $\sqcup$, $\setminus$, and $\neg$ are not the same as the standard set operations. The following relationship holds between the standard set intersection and the interval-set intersection:

$$[A_l, A_u] \cap [B_l, B_u] \subseteq [A_l, A_u] \sqcap [B_l, B_u]. \qquad (10)$$

However, there do not exist similar relationships between other set and interval-set operations. When only degenerate interval sets are used, the interval-set operations $\sqcap$, $\sqcup$ and $\setminus$ reduce to the usual set intersection, union and difference.

4. Interval-Set Algebras

We study two types of algebras based on two ordering relations defined on the family of interval sets, one is called inclusion or truth ordering and the other is called knowledge ordering [15, 36].

4.1. Algebra based on inclusion ordering

Suppose A and B are two sets representing extensions of two partially known concepts. They can be expressed as interval sets $[A_l, A_u]$ and $[B_l, B_u]$, respectively. Although we may not know the exact members of A and B, we may have additional information suggesting $A \subseteq B$. Based on interval-set representation, it is reasonable to require that $A_l \subseteq B_l$ and $A_u \subseteq B_u$. This leads to the introduction of an inclusion ordering $\sqsubseteq$ on interval sets[7, 15, 36, 42].

The inclusion of interval sets may be defined by:

$$\begin{aligned} & [A_l, A_u] \sqsubseteq [B_l, B_u] \\ \Longleftrightarrow\ & A_l \subseteq B_l \wedge A_u \subseteq B_u \\ \Longleftrightarrow\ & (\forall A \in [A_l, A_u]\ \exists B \in [B_l, B_u]\ A \subseteq B) \wedge \\ & (\forall B \in [B_l, B_u]\ \exists A \in [A_l, A_u]\ A \subseteq B). \qquad (11) \end{aligned}$$

Based on this definition, for two interval sets $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A} = \mathscr{B}$ if and only if $\mathscr{A} \sqsubseteq \mathscr{B}$ and $\mathscr{B} \sqsubseteq \mathscr{A}$.

For $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \in I(2^U)$, the following properties hold for the $\sqsubseteq$ relation:

(i1) $\mathscr{A} \sqsubseteq \mathscr{B} \Longleftrightarrow \mathscr{A} \sqcap \mathscr{B} = \mathscr{A}$,
$\mathscr{A} \sqsubseteq \mathscr{B} \Longleftrightarrow \mathscr{A} \sqcup \mathscr{B} = \mathscr{B}$;

(i2) $\mathscr{A} \sqsubseteq \mathscr{B}$ and $\mathscr{C} \sqsubseteq \mathscr{D} \Longrightarrow \mathscr{A} \sqcap \mathscr{C} \sqsubseteq \mathscr{B} \sqcap \mathscr{D}$,
$\mathscr{A} \sqsubseteq \mathscr{B}$ and $\mathscr{C} \sqsubseteq \mathscr{D} \Longrightarrow \mathscr{A} \sqcup \mathscr{C} \sqsubseteq \mathscr{B} \sqcup \mathscr{D}$;

(i3) $\mathscr{A} \sqcap \mathscr{B} \sqsubseteq \mathscr{A}$, $\quad \mathscr{A} \sqcap \mathscr{B} \sqsubseteq \mathscr{B}$,
$\mathscr{A} \sqsubseteq \mathscr{A} \sqcup \mathscr{B}$, $\quad \mathscr{B} \sqsubseteq \mathscr{A} \sqcup \mathscr{B}$.

Like its counterpart in set theory, the relation $\sqsubseteq$ on $I(2^U)$ is a reflexive and transitive relation. On the other hand, for two interval sets $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{A} \sqsubseteq \mathscr{B}$, the difference $\mathscr{A} \setminus \mathscr{B}$ is not necessarily equal to $[\emptyset, \emptyset]$. In this case, we only have $\mathscr{A} \sqsubseteq \mathscr{B} \Longrightarrow \emptyset \in \mathscr{A} \setminus \mathscr{B}$.

The inclusion relation $\sqsubseteq$ is the ordering relation that defines the lattice with operations $\sqcap$ and $\sqcup$. We obtain two interval-set algebras. One algebra is a completely distributive lattice $(I(2^U), \sqcap, \sqcup)$, or $(I(2^U), \sqsubseteq)$, consisting of all possible interval sets. The other is a completely distributive lattice $(IR(2^U), \sqcap, \sqcup)$, or $(IR(2^U), \sqsubseteq)$, consisting of all interval sets whose ending sets are from an σ-algebra $\sigma(U/E)$.

4.2. Algebra based on knowledge ordering

An interval set is a family of sets. We can apply standard set operations and relations to interval sets. Consider two interval sets with $[A_l, A_u] \subseteq [B_l, B_u]$. It is reasonable to say that we have more knowledge about the concept represented by $[A_l, A_u]$ than the concept represented by $[B_l, B_u]$, as we are more sure about the former than the latter. This suggests that the standard set inclusion provides a knowledge ordering on interval sets [15, 36].

A knowledge ordering $\preceq_k$ on interval sets can be defined by [15]:

$$\begin{aligned} & [B_l, B_u] \preceq_k [A_l, A_u] \\ \Longleftrightarrow\ & [A_l, A_u] \subseteq [B_l, B_u] \\ \Longleftrightarrow\ & B_l \subseteq A_l \subseteq A_u \subseteq B_u. \qquad (12) \end{aligned}$$

In some sense, the knowledge ordering reflect the fact that $[A_l, A_u]$ is tighter than $[B_l, B_u]$. Again, two interval sets are equal, namely, $\mathscr{A} = \mathscr{B}$, if and only if $A_l = B_l$ and $A_u = B_u$. For easy interpretation, we will use the standard set inclusion $\subseteq$ as a knowledge ordering.

The set intersection of two interval sets is an interval set, namely, for $\mathscr{A} = [A_l, A_u]$ and $\mathscr{B} = [B_l, B_u]$,

$$\mathscr{A} \cap \mathscr{B} = \begin{cases} [A_l \cup B_l, A_u \cap B_u], & A_l \cup B_l \subseteq A_u \cap B_u; \\ [\emptyset, \emptyset], & otherwise. \end{cases} \qquad (13)$$

However, $\mathscr{A} \cup \mathscr{B}$ is not necessarily an interval set in general; it is an interval set when $\mathscr{A} \subseteq \mathscr{B}$ or $\mathscr{B} \subseteq \mathscr{A}$. For $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \in I(2^U)$, the following properties hold for the $\subseteq$ relation on interval sets:

(k1) $\mathscr{A} \subseteq \mathscr{B} \Longleftrightarrow \mathscr{A} \cap \mathscr{B} = \mathscr{A}$,
$\mathscr{A} \subseteq \mathscr{B} \Longleftrightarrow \mathscr{A} \cup \mathscr{B} = \mathscr{B}$;

(k2) $\mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{C} \subseteq \mathscr{D} \Longrightarrow \mathscr{A} \cap \mathscr{C} \subseteq \mathscr{B} \cap \mathscr{D}$;

(k3) $\mathscr{A} \cap \mathscr{B} \subseteq \mathscr{A}$, $\quad \mathscr{A} \cap \mathscr{B} \subseteq \mathscr{B}$.

In these properties, we only consider a standard set operation on interval sets when the result is also an interval set.

The relation $\subseteq$ on $I(2^U)$ is a reflexive and transitive relation. It is the ordering relation for defining the semi-lattice operation $\cap$. With respect to the knowledge ordering $\subseteq$, we again have two interval-set algebras. One algebra is the semi-lattice $(I(2^U), \cap)$, or $(I(2^U), \subseteq)$, and the other algebra is the semi-lattice $(IR(2^U), \cap)$, or $(IR(2^U), \subseteq)$.

5. Studies Related to Interval Sets

In this section, we briefly review additional studies related to interval sets in order to further demonstrate the potential of interval sets and interval-set algebras.

Interval sets and rough sets: Pawlak rough sets, Iwiński rough sets and interval sets are related to pairs of definable sets. Many studies treat Iwiński rough sets as if they are Pawlak rough sets by ignoring their subtle differences. Pawlak rough-set operations are non-truth-functional and Iwiński rough-set operations are truth-functions [44, 57].

Many studies on rough sets are related to interval sets. Davvaz [5] proposed T-rough sets based on set-valued mapping. They are Iwiński rough sets, or interval sets, constructed based on two universes. Many authors, including, for example, Wong *et al.* [37], Yao [44], Pei and Xu [28], Shu and He [31], Li [10], Li and Zhang [11], Gong and Sun [6], Mi *et al.* [16] and Pei and Fan [27], studied rough sets and fuzzy rough sets over universes. The results can be easily related to interval sets and interval fuzzy sets [52]. Yamaguchi *et al.* [38] proposed the notion of grey-rough sets based on interval-valued attributes and compared it with other definitions of rough sets and interval sets. Oukbir [21] examined the connections between rough sets and interval sets and considered their potential applications in spatial information systems. Kerre [8] examined connections among flou sets, fuzzy sets, and rough sets.

Mousavi and Jabedar-Maralani [19] examined the relationships between relative sets and interval sets. The relative sets consider both inclusion (truth) ordering and knowledge ordering of a bilattice. They pointed out that the original interval-set algebra considers only the inclusion ordering but not the knowledge ordering. The studies by Marek and Truszczyński [15] and by Wolski [36] consider both orderings. In the Section 4.2 of this paper, we also consider the knowledge ordering, although slightly different from these studies. Wang and Zhang [35] investigated two implication operators on interval sets and rough sets and studied other structures of interval-set algebras. Zhang and Jia [58] proposed the notion of lattice-valued interval sets and examined the differences between flou sets and interval sets.

Interval sets and granular computing: Granular computing is a new paradigm of human-inspired computing [12, 26, 40, 49]. The triarchic theory of granular computing consists of the structured thinking, structured problem solving and structured information processing with multi-level granular structures [46, 48]. A central notion of granular computing is a granule that presents an entity or a focal point under discussion. Rough set theory [23, 24], quotient space theory [56] and fuzzy set theory [54, 55] provide concrete models of granular computing [45]. Interval-number algebra and interval-set algebra are also concrete models of granular computing [45]. Tahayori *et al.* [33] argue that distributed intervals can be used to establish a formal framework for information granulation, where interval-set algebra is a special case.

Interval sets, three-valued logic, and reasoning: As suggested by Yao [42], interval sets provide another qualitative sets in comparison with fuzzy sets. An interval set can be characterized by a three-valued membership function from U to the three-elements set $\{0, u, 1\}$, where 0 stands for a non-membership, 1 for membership, and u for an unknown membership that may actually be either 0 or 1. For an interval set $[A_l, A_u]$, elements in A_l have the membership value 1, elements in $A_u - U_l$ have the membership value u, and elements in $U - A_u$ have membership value 0. The same membership function can be used to interpret a flou set [8]. Under this setting, interval-set operations are defined by Kleene's three-valued logic [50]. Yao [44] showed that interval sets and operations are related to Iwiński rough sets and operations. Dai [3, 4] discussed connections between rough sets and 3-valued Łukasiewicz algebras based essentially on Iwiński rough sets. Reasoning based on interval sets have been examined by several authors (for example, Yao [43], Yao and Li [50], Yao and Wong [53], Yamauchi and Mukaidono [39], Qi [29], and Qi *et al.* [30]).

With respect to the same mathematical structure used, we have very different semantics interpretations. A flou set is a set defined by a three-valued membership function, and an interval set is a family of sets.

Interval set clustering: Many clustering approaches treat clusters as sets with well-defined boundary or fuzzy sets with quantitatively gradually changing boundary. Motivated by the notions of lower and upper approximations in rough sets, Lingras [13, 14] proposed and studied rough set clustering and interval set clustering. A basic assumption is that the actual set corresponding to a cluster is not entirely known. For each cluster, it is only possible to provide a pair of lower and upper bounds based on the available information. This assumption seems to be consistent with the interpretation of interval sets as representations of partially known concepts. The extensive results from Lingras' group demonstrate that interval set clustering is applicable to real-world problems.

Interval-set-valued information tables and analysis: In an interval-set-valued information table, an object may have an interval set as its value on an attribute. This presents a generalization of set-valued information tables. Yao and Liu [51] introduced a generalized decision logic for interval-set-valued information tables.

Leung *et al.* [9] and Miao *et al.* [17] considered the problem of attribute reduction and rule learning in interval-valued information tables. Sun *et al.* [32] considered attribute reduction in interval-valued fuzzy information tables. The results may be applied to attribute reduction in interval-set-valued information tables.

6. Conclusion

Interval sets provide a new means for representing partially known concepts or for approximating undefinable concepts or complex concepts. Interval sets are closely related to, and complementary to, fuzzy sets, rough sets, and flou sets. An important characteristic of interval sets is that an interval set is a family of sets. Based on this interpretation, operations on interval sets are investigated. Based on an inclusion ordering and a knowledge ordering, two types of interval-set algebras are examined. A brief summary of studied related to interval sets is provided.

By studying interval sets and interval-set algebras, we hope to gain more insights into the representation and processing of imprecise or partially known concepts, and into the approximations of undefinable or complex concepts. One would find more development and applications of interval sets in the near future.

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