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Data Analysis Based on Discernibility and Indiscernibility

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Abstract

Rough set theory models similarities and differences of objects based on the notions of indiscernibility and discernibility. With respect to any subset of attributes, one can define two pairs of dual relations: the strong indiscernibility and weak discernibility relations, and the weak indiscernibility and strong discernibility relations. The similarities of objects are examined by the indiscernibility relations, and the difference by the discernibility relations, respectively. Alternatively, one can construct an indiscernibility matrix to represent the family of strong indiscernibility or weak discernibility relations. One also can construct a discernibility matrix to represent the family of strong discernibility or weak indiscernibility relations. The consideration of the matrix-counterpart of relations, and the relation-counterpart of matrices, brings more insights into rough set theory.

Based on indiscernibility and discernibility, three different types of reducts can be constructed, keeping the indiscernibility, discernibility, and indiscernibility-and-discernibility relations, respectively. Although the indiscernibility reducts have been intensively studied in the literature, the other two types of reducts are relatively new and require more attention. The existing methods for constructing the indiscernibility reducts also can be applied to construct the other two types of reducts. An empirical experiment for letter recognition is reported for demonstrating the usefulness of the discussed relations and reducts.

Key words: Rough sets, indiscernibility and discernibility relations, indiscernibility and discernibility matrices, reducts

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1 Introduction

The dual notions of similarities and differences play a crucial role in fields such as concept formation, machine learning, data mining, data analysis, cluster analysis, and many more [25–27]. The similarities of objects lead naturally to their grouping and integration, and the differences lead to group division and decomposition. It is important to extract similarities of objects by ignoring certain differences in order to form a useful cluster or a high-level concept, and to identify differences among a set of similar objects in order to form sub-concepts. A fundamental task of intelligent data analysis is to make explicit patterns and knowledge embedded in data through an investigation of similarities and differences of data items.

One can study either the similarities or the differences among objects. By focusing on the similarities, one can differentiate or discern objects. By focusing on the difference, one can summarize common properties and characters of objects [18,25,29]. The study of similarity and difference can find many real-life applications. For example, in social science or politics, one can emphasize the differences between entities and thus virtually enlarge and aggravate the conflicts and discordance; or, emphasize the commonness and therefore create a concordant atmosphere for negotiation and communication [15].

The theory of rough sets, as a theory of data analysis, models similarities and differences of objects based on the notions of indiscernibility and discernibility. There are two fundamental issues: representations of indiscernibility and discernibility, and attribute reduction (information table simplification) based on indiscernibility and discernibility.

1.1 Representations of indiscernibility and discernibility

Suppose a finite set of objects is described by a finite set of attributes. With respect to any subset of attributes, one can define a pair of dual indiscernibility and discernibility relations [12,13]. Two objects are considered to be indiscernible or equivalent if and only if they have the same values for *all* attributes in the set. In other words, in terms of the given set of attributes, it is impossible to differentiate the two objects. As a dual relation to indiscernibility, two objects are considered to be discernible if and only if they have different values for *at least one* attribute. Since the pair of indiscernibility and discernibility relations are defined with respect to the set of all attributes and at least one attribute, respectively, they may be viewed as strong indiscernibility and weak discernibility. The strong indiscernibility is indeed the strongest type of similarity between objects and is characterized by an equivalent relation (i.e., reflexive, symmetric and transitive). For an easy discussion, we also refer to a strong indiscernibility relation as an indiscernibility relation. The weak discernibility can capture the weakest differences between objects. Their duality suggests that two objects are either strongly indiscernible or weakly discernible.

In contrast to strong indiscernibility and weak discernibility, we can define a pair of dual weak indiscernibility and strong discernibility. Two objects are weakly indiscernible if and only if they have the same value for *at least one* attribute. They are strongly discernible if and only if they have different values for *all* attributes. For simplicity, we also refer to a strong discernibility relation as a discernibility relation. By their duality, a pair of objects is either weakly indiscernible or strongly discernible. The weak indiscernibility relation is a compatible or tolerance relation (i.e., reflexive and symmetric).

The pair of weak and strong indiscernibility relations represent the two extreme cases of similarity, and the pair of weak and strong discernibility relations represent the two extreme cases of difference. They in fact reflect the qualitative nature of data. There are two possible directions for generalizing indiscernibility and discernibility into a quantitative framework. One direction deals with graded indiscernibility or discernibility, where one can count the number of attributes on which two objects have the same values or different values. In the other direction, instead of using the trivial quality relation, one can use a distance or similarity function to quantify the closeness of two attribute values. Examples of such studies include valued-similarity and tolerance [17,25,26,35], neighborhood systems [11,33,34], rough inclusion [16,20], and many more.

Alternatively, indiscernibility and discernibility can be represented using matrices. For discernibility, we have a discernibility matrix whose rows and columns correspond to the set of objects [21]. The cell corresponding to a pair of objects consists of all those attributes on which the two objects have different values. One can easily establish a connection between the strong and weak discernibility relations and the discernibility matrix. More specifically, two objects are weakly discernible if their corresponding cell in the discernibility matrix is non-empty. They are strongly discernible with respect to a set of attributes if the set is a subset of the corresponding cell of the discernibility matrix. For indiscernibility, we can have an indiscernibility matrix, with each cell consists of all the attributes on which the two objects have the same values. One can establish a connection between the strong and weak indiscernibility relations and the indiscernibility matrix. Two objects are weakly indiscernible if their corresponding cell in the indiscernibility matrix is non-empty. They are strongly indiscernible with respect to a set of attributes if the set is a subset of the corresponding cell of the indiscernibility matrix.

1.2 Attribute reduction

The concept of a reduct is a fundamental notion supporting rough set based data analysis [2,8,13,14,19,21,23,30,31,36–38]. As the result of an attribute reduction process, a reduct is a minimum set of attributes, which is as informative as the original set of attributes. Specifically, a reduct is a subset of attributes that is jointly sufficient and individually necessary for preserving the same information or property as that is provided by the entire set of attributes. In this paper, we consider three types of reducts: indiscernibility reducts that preserve indiscernibility, discernibility reducts that preserve discernibility, and indiscernibility-and-discernibility reducts that preserve both indiscernibility and discernibility.

An indiscernibility reduct contains a minimum set of attributes such that any two objects indiscernible with respect to the original set of attributes are also indiscernible with respect to the reduct. By preserving the strong indiscernibility, we place emphasis on separating objects based on their differences. A discernibility reduct also contains a minimum set of attributes such that any two objects discernible with respect to the original set of attributes also are discernible with respect to the reduct. Indiscernibility reduct construction methods can be formulated based on either the indiscernibility relation or the weak discernibility relation. Discernibility reduct construction methods can be formulated based on either the discernibility relation or the weak indiscernibility relation.

1.3 Overview of the paper

The existing study of reduct construction demonstrates two features. First, there are two distinct groups characterized by applying two different sets of notations. One group focuses on the indiscernibility relation in a universe that captures the equivalence of objects [3,13,14,27,30]. The other group focuses on the discernibility matrix that explores the differences of objects [21,22,31]. A *discernibility function* defined for a discernibility matrix reveals the weak discernibility relation between two objects, i.e., we can distinguish two corresponding objects by any one attribute in a matrix element. Since the notations seem much different, the connection between these two groups is weak. Second, since the indiscernibility and the weak discernibility are two dual relations, the constructed results of both groups are the indiscernibility reducts of the entire attribute set. The discernibility reducts, as well as the indiscernibility-and-discernibility reducts are less studied.

Owe to these two features, the contribution of this paper is two-fold. First, we suggest that based on indiscernibility and discernibility, rough set based data analysis can be unified into one model. That is, since these two pairs of relations are complementary, they can be formulated based on the same notion. Alternatively, the family of relations can be effectively expressed by a matrix. Based on the two matrices, two dual matrix functions can be studied. Second, three different kinds of reducts based on indiscernibility and discernibility are explored. The complementary study brings new insights into data analysis approaches, explores different aspects of data understanding, different angles of data summarizations and descriptions, and different types of discovered knowledge. The variety of views can thus satisfy a wider range of needs of different users.

2 Indiscernibility and Discernibility Relations

Information tables, also known as information systems, data tables, attribute-value systems, are investigated by many researchers of rough set theory [10,13,14,30]. It is assumed that data are represented in a table form, where a set of objects (rows) are described by a finite set of attributes (columns).

Definition 1 *An information table S is the tuple*

$$S = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}),$$

where U is a finite nonempty set of objects called universe, At is a finite nonempty set of attributes, V_a is a nonempty set of values for an attribute $a \in At$, and $I_a : U \longrightarrow V_a$ is an information function, such that for an object $x \in U$, an attribute $a \in At$, and a value $v \in V_a$, $I_a(x) = v$ means that the object x has the value v on the attribute a.

2.1 Qualitative indiscernibility and discernibility relations

Given a subset of attributes $A \subseteq At$, four binary relations between objects can be differentiated in an information table.

Definition 2 *Given a subset of attributes $A \subseteq At$, four relations on U are defined by:*

$$\begin{aligned}
\mathrm{IND}(A) &= \{(x, y) \in U \times U \mid \forall a \in A, I_a(x) = I_a(y)\},\\
\mathrm{WIND}(A) &= \{(x, y) \in U \times U \mid \exists a \in A, I_a(x) = I_a(y)\},\\
\mathrm{DIS}(A) &= \{(x, y) \in U \times U \mid \forall a \in A, I_a(x) \neq I_a(y)\},
\end{aligned}$$

$$\mathrm{WDIS}(A) = \{(x, y) \in U \times U \mid \exists a \in A, I_a(x) \neq I_a(y)\}.$$

A strong indiscernibility relation with respect to A is denoted as IND(A). Two objects in U satisfy IND(A) if and only if they have the same values on all attributes in A. It can be verified that an indiscernibility relation is reflexive, symmetric and transitive, namely, it is an equivalence relation. It partitions the universe U into a family of disjoint subsets, denoted as $U/\mathrm{IND}(A) = \{[x]_{\mathrm{IND}(A)} \mid x \in U\}$, where $[x]_{\mathrm{IND}(A)}$ is the A-definable equivalence class containing x, i.e.,

$$[x]_{\mathrm{IND}(A)} = \{y \in U \mid (x, y) \in \mathrm{IND}(A)\}.$$

On the other extreme, a weak indiscernibility relation WIND(A) with respect to A only requires that two objects have the same value on at least one attribute in A. A weak indiscernibility relation is reflexive, symmetric, but not necessarily transitive. Such a relation is known as a compatibility or a tolerance relation. The two types of relations are studied extensively in rough set theory for different types of approximation spaces [4,5,17,26,27,29,30],

As the complement of a strong indiscernibility relation, a weak discernibility relation WDIS(A) states that two objects are discernible if and only if they have different values on at least one attribute in A. A weak discernibility relation is irreflexive and symmetric, but not transitive. The complement of a weak indiscernibility relation is a strong discernibility relation DIS(A), states that two objects are strongly discernible with respect to A if they have different values on all attributes in A. A strong discernibility relation is irreflexive and symmetric, but not transitive. The strong and weak discernibility relations are also called as the strong and weak diversity relations [12].

For a singleton attribute set $\{a\}$, the strong and weak indiscernibility relations are the same, and the strong and weak discernibility relations are the same. That is,

$$\mathrm{IND}(\{a\}) = \mathrm{WIND}(\{a\});$$
$$\mathrm{DIS}(\{a\}) = \mathrm{WDIS}(\{a\}).$$

For a subset of attributes, the indiscernibility and discernibility relations can be conveniently expressed in terms of the corresponding relations defined by singleton attribute subsets:

$$\mathrm{IND}(A) = \bigcap_{a \in A} \mathrm{IND}(\{a\});$$

$$\begin{aligned}
\mathrm{WIND}(A) &= \bigcup_{a\in A} \mathrm{IND}(\{a\});\\
\mathrm{DIS}(A) &= \bigcap_{a\in A} \mathrm{DIS}(\{a\});\\
\mathrm{WDIS}(A) &= \bigcup_{a\in A} \mathrm{DIS}(\{a\}).
\end{aligned}$$

It immediately follows that the four relations are related together. First, the weak and strong versions are related by a subset relationship:

$$\begin{aligned}
&(\mathrm{S1}).\quad \mathrm{IND}(A) \subseteq \mathrm{WIND}(A);\\
&(\mathrm{S2}).\quad \mathrm{DIS}(A) \subseteq \mathrm{WDIS}(A).
\end{aligned}$$

This is, if two objects are strongly indiscernible/discernible, they are weakly indiscernible/discernible. An indiscernibility relation is a subset of the weak indiscernibility relation defined by the same attribute set. Similarly, a discernibility relation is a subset of the weak discernibility relation defined by the same attribute set.

Second, the indiscernibility and discernibility relations are related by a complementary relationship:

$$\begin{aligned}
&(\mathrm{C1}).\quad \mathrm{WDIS}(A) = \mathrm{IND}^c(A);\\
&(\mathrm{C2}).\quad \mathrm{WIND}(A) = \mathrm{DIS}^c(A),
\end{aligned}$$

where for a relation $\mathcal{R} \subseteq U \times U$, $\mathcal{R}^c = U \times U - \mathcal{R}$ denotes its complement. Therefore, we have two pairs of complementary relations, the pair $(\mathrm{IND}(A), \mathrm{WDIS}(A))$ of the strong indiscernibility relation and the weak discernibility relation, and the pair $(\mathrm{DIS}(A), \mathrm{WIND}(A))$ of the strong discernibility relation and the weak indiscernibility relation.

For two subsets of attributes $B \subseteq A \subseteq At$, their induced indiscernibility and discernibility relations satisfy the following monotocity with respect to set inclusion:

$$\begin{aligned}
&B \subseteq A \Longrightarrow \mathrm{IND}(A) \subseteq \mathrm{IND}(B);\\
&B \subseteq A \Longrightarrow \mathrm{WIND}(B) \subseteq \mathrm{WIND}(A);\\
&B \subseteq A \Longrightarrow \mathrm{DIS}(A) \subseteq \mathrm{DIS}(B);\\
&B \subseteq A \Longrightarrow \mathrm{WDIS}(B) \subseteq \mathrm{WDIS}(A).
\end{aligned}$$

That is, the strong indiscernibility/discernibility relations are monotonically decreasing, and the weak indiscernibility/discernibility relations are monotonically increasing.

2.2 Quantitative indiscernibility and discernibility relations

The strong and weak indiscernibility relations represent the two extreme points, which bound many levels of indiscernibility. With respect to a non-empty set of attributes $A \subseteq At$, a graded, or quantitative, indiscernibility relation is defined as a mapping from $U \times U$ to the unit interval $[0, 1]$.

Definition 3 *Given a subset of attributes $A \subseteq At$ and a pair of objects $(x, y) \in U \times U$, the quantitative indiscernibility relation* $\mathrm{ind}(A)(x, y)$ *is defined by:*

$$\mathrm{ind}(A)(x, y) = \frac{|\{a \in A \mid I_a(x) = I_a(y)\}|}{|A|},$$

where $|.|$ denotes the cardinality of a set.

A quantitative indiscernibility relation satisfies the following properties:

(i1). $\mathrm{ind}(A)(x, x) = 1$,
(i2). $\mathrm{ind}(A)(x, y) = \mathrm{ind}(A)(y, x)$.

The properties (i1) and (i2) reflect that a quantitative indiscernibility relation is reflexive and symmetric.

The quantitative indiscernibility relations are connected to the strong and weak indiscernibility relations as follows:

$$\begin{aligned} x\mathrm{IND}(A)y &\Longleftrightarrow \mathrm{ind}(A)(x, y) = 1, \\ x\mathrm{WIND}(A)y &\Longleftrightarrow \mathrm{ind}(A)(x, y) > 0. \end{aligned}$$

The quantitative indiscernibility relation can be regarded as a fuzzy relation, thus the strong and weak indiscernibility relations are in fact the core and the support of the fuzzy relation. In general, we can apply the notion of α-cut of fuzzy sets to obtain different quantitative indiscernibility relations. An α-cut quantitative indiscernibility relation $\mathrm{ind}_\alpha(A)$ is a binary relations between objects, defined by:

$$\mathrm{ind}_\alpha(A) = \{(x, y) \in U \times U \mid \mathrm{ind}(A)(x, y) \geq \alpha\}.$$

If the object pair (x, y) satisfies $\mathrm{ind}_\alpha(A)$, then x and y are indiscernible with respect to α. The relation $\mathrm{ind}_\alpha(A)$ induces a covering of the universe U, denoted as $U/\text{ind}_\alpha(A) = \{(x)_{\text{ind}_\alpha(A)} \mid x \in U\}$, where $(x)_{\text{ind}_\alpha(A)}$ is the quasi-equivalence class containing x, i.e.,

$$(x)_{\text{ind}_\alpha(A)} = \{y \in U \mid (x, y) \in \text{ind}_\alpha(A)\}.$$

The quasi-equivalence classes are flexible granules. They possibly overlap each other. The research on quantitative indiscernibility relations has led to the study based on blocks [4,5], templates [1,9], rough inclusion [16,20], and tolerance, similarity or neighborhood relations [11,33,34]

Definition 4 *Given a subset of attribute $A \subseteq At$ and a pair of objects $(x, y) \in U \times U$, the quantitative discernibility relation* $\text{dis}(A)(x, y)$ *is defined as the complement of a quantitative indiscernibility relation:*

$$\begin{aligned} \text{dis}(A)(x, y) &= 1 - \text{ind}(A)(x, y) \\ &= \frac{|\{a \in A \mid I_a(x) \neq I_a(y)\}|}{|A|}. \end{aligned}$$

A quantitative indiscernibility relation satisfies the following properties:

(d1). $\text{dis}(A)(x, x) = 0$,
(d2). $\text{dis}(A)(x, y) = \text{dis}(A)(y, x)$.

The properties (d1) and (d2) show that a quantitative discernibility relation is irreflexive and symmetric.

The qualitative discernibility relations are the two extremes of quantitative relations, namely,

$$\begin{aligned} x\text{DIS}(A)y &\iff \text{dis}(A)(x, y) = 1, \\ x\text{WDIS}(A)y &\iff \text{dis}(A)(x, y) > 0. \end{aligned}$$

In general, we also can apply the notion of α-cut of fuzzy sets to obtain different quantitative discernibility relations. An α-cut quantitative discernibility relation $\text{dis}_\alpha(A)$ also is a binary relations between objects, defined by:

$$\text{dis}_\alpha(A) = \{(x, y) \in U \times U \mid \text{dis}(A)(x, y) \geq \alpha\}.$$

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Table 1
An information table.

U	At					
	a	b	c	d	e	f
o_1	1	1	1	1	1	1
o_2	1	0	1	0	1	1
o_3	0	0	1	1	0	0
o_4	1	1	1	0	0	1
o_5	1	0	1	0	1	1
o_6	0	0	0	1	1	0
o_7	1	0	1	1	1	1
o_8	0	0	0	0	1	1
o_9	1	0	0	1	0	0

If the object pair (x, y) satisfies $\mathrm{dis}_\alpha(A)$, then x and y are discernible with respect to α.

Example 1 *Table 1 is an information table with nine objects and six attributes. We will use this table for the rest of the paper.*

For the information Table 1, we have, for example, the following partitions defined by attribute sets $\{a\}$, $\{b\}$ and $\{a, b\}$, respectively:

$$\begin{aligned} U/\mathrm{IND}(\{a\}) &= \{\{o_1, o_2, o_4, o_5, o_7, o_9\}, \{o_3, o_6, o_8\}\}; \\ U/\mathrm{IND}(\{b\}) &= \{\{o_1, o_4\}, \{o_2, o_3, o_5, o_6, o_7, o_8, o_9\}\}; \\ U/\mathrm{IND}(\{a, b\}) &= \{\{o_1, o_4\}, \{o_2, o_5, o_7, o_9\}, \{o_3, o_6, o_8\}. \end{aligned}$$

Examples of the strong and weak indiscernibility/discernibility relations defined by the three attribute sets are illustrated in Figure 1, where 1 means that the two corresponding objects are related according to the relation.

Given an information table, we can obtain both the indiscernibility and discernibility relations with respect to a subset of attributes. On the other hand, given all the indiscernibility and discernibility relations, we cannot recover the original information table. A relation only tells whether two objects are discernible or indiscernible with respect to the attribute set, but does not keep the attribute values.

IND($\{a\}$)= WIND($\{a\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1	1	1		1	1		1		1
o_2	1	1		1	1		1		1
o_3			1			1			1
o_4	1	1		1	1		1		1
o_5	1	1		1	1		1		1
o_6			1			1		1	
o_7	1	1		1	1		1		1
o_8						1		1	
o_9	1	1	1	1	1		1		1

DIS($\{a\}$)= WDIS($\{a\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1			1			1		1	
o_2			1			1		1	
o_3	1	1		1	1		1	1	
o_4			1			1		1	
o_5			1			1		1	
o_6	1	1		1	1		1		1
o_7			1			1		1	
o_8	1	1	1	1	1		1		1
o_9						1		1	

IND($\{b\}$)= WIND($\{b\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1	1			1					
o_2		1	1		1	1	1	1	1
o_3		1	1		1	1	1	1	1
o_4	1			1					
o_5		1	1		1	1	1	1	1
o_6		1	1		1	1	1	1	1
o_7		1	1		1	1	1	1	1
o_8		1	1		1	1	1	1	1
o_9		1	1		1	1	1	1	1

DIS($\{b\}$)= WDIS($\{b\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1		1	1		1	1	1	1	1
o_2	1			1					
o_3	1			1					
o_4		1	1		1	1	1	1	1
o_5	1			1					
o_6	1			1					
o_7	1			1					
o_8	1			1					
o_9	1			1					

IND($\{a,b\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1	1			1					
o_2		1			1		1		1
o_3			1			1		1	
o_4	1			1					
o_5		1			1		1		1
o_6			1			1		1	
o_7		1			1		1		1
o_8			1			1		1	
o_9		1			1		1		1

DIS($\{a,b\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1			1			1		1	
o_2									
o_3	1			1					
o_4			1			1		1	
o_5									
o_6	1			1					
o_7									
o_8	1			1					
o_9									

WIND($\{a,b\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1	1	1		1	1		1		1
o_2	1	1	1	1	1	1	1	1	1
o_3		1	1		1	1	1	1	1
o_4	1	1		1	1		1		1
o_5	1	1	1	1	1	1	1	1	1
o_6		1	1		1	1	1	1	1
o_7	1	1	1	1	1	1	1	1	1
o_8		1	1		1	1	1	1	1
o_9	1	1	1	1	1	1	1	1	1

WDIS($\{a,b\}$)	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1		1	1		1	1	1	1	1
o_2	1		1	1		1		1	
o_3	1	1		1	1		1		1
o_4		1	1		1	1	1	1	1
o_5	1		1	1		1		1	
o_6	1	1		1	1		1		1
o_7	1		1	1		1		1	
o_8	1	1		1	1		1		1
o_9	1		1	1		1		1	

Fig. 1. Examples of the indiscernibility and discernibility relations

It should be noted that the four qualitative relations and the two quantitative relations defined in this section are based on the information function I_a and a simple assumption that each object x has one and only one value on the attribute a. In real world cases, data are generally imprecise, tend to be noisy, and values for attributes are often missing [22]. For example, we may have some objects that have missing values on some attributes, or have multiple values on some other attributes. In these uncertain cases, different generalizations need to be considered [5,19].

3 Indiscernibility and Discernibility Matrices

The relationships between objects can be alternatively expressed as matrices. For example, an indiscernibility matrix can be defined as follows.

Definition 5 *Given an information table S, its indiscernibility matrix* **im** *is a $|U| \times |U|$ matrix with each element* $\mathbf{im}(x, y)$ *defined as:*

$$\mathbf{im}(x, y) = \{a \in At \mid I_a(x) = I_a(y), x, y \in U\}.$$

Each cell of an indiscernibility matrix stores the attributes on which the corresponding two objects have the same values. The indiscernibility matrix **im** is symmetric, i.e., $\mathbf{im}(x, y) = \mathbf{im}(y, x)$, and $\mathbf{im}(x, x) = At$.

In contrast to an indiscernibility matrix, a discernibility matrix can be defined as follows [21].

Definition 6 *Given an information table S, its discernibility matrix* **dm** *is a $|U| \times |U|$ matrix with each element* $\mathbf{dm}(x, y)$ *defined as:*

$$\mathbf{dm}(x, y) = \{a \in At \mid I_a(x) \neq I_a(y), x, y \in U\}.$$

Each element of a discernibility matrix stores the attributes on which the corresponding two objects have distinct values. The discernibility matrix **dm** is symmetric, i.e., $\mathbf{dm}(x, y) = \mathbf{dm}(y, x)$, and $\mathbf{dm}(x, x) = \emptyset$. By definition, the two matrices are complementary, i.e., for any object pair $(x, y) \in U \times U$, $\mathbf{im}(x, y) = (\mathbf{dm}(x, y))^c = At - \mathbf{dm}(x, y)$.

Similar to the case of relations, given an information table, we can obtain its indiscernibility matrix and discernibility matrix. Conversely, given an indiscernibility matrix or a discernibility matrix, we cannot recover its information table. Each element of a matrix keeps only the names of attributes whose values are the same, or different, for two objects, but not the values of those attributes.

There is a close connection between a matrix and its corresponding strong and weak relations.

Theorem 1 *Suppose A is a subset of attributes. The indiscernibility and discernibility relations and the indiscernibility and discernibility matrices can be defined by each other as follows:*

$$
\begin{aligned}
\mathrm{IND}(A) &= \{(x,y) \in U \times U \mid A \subseteq \mathbf{im}(x,y)\}, \\
&= \{(x,y) \in U \times U \mid A \cap \mathbf{dm}(x,y) \neq \emptyset\}; \\
\mathrm{WIND}(A) &= \{(x,y) \in U \times U \mid A \cap \mathbf{im}(x,y) \neq \emptyset\}, \\
&= \{(x,y) \in U \times U \mid A \nsubseteq \mathbf{dm}(x,y)\}; \\
\mathrm{DIS}(A) &= \{(x,y) \in U \times U \mid A \subseteq \mathbf{dm}(x,y)\}, \\
&= \{(x,y) \in U \times U \mid A \cap \mathbf{im}(x,y) \neq \emptyset\}; \\
\mathrm{WDIS}(A) &= \{(x,y) \in U \times U \mid A \cap \mathbf{dm}(x,y) \neq \emptyset\}, \\
&= \{(x,y) \in U \times U \mid A \nsubseteq \mathbf{im}(x,y)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{im}(x,y) &= \{a \in At \mid x\mathrm{IND}(\{a\})y,\ (x,y) \in U \times U\}; \\
\mathbf{dm}(x,y) &= \{a \in At \mid x\mathrm{DIS}(\{a\})y,\ (x,y) \in U \times U\}.
\end{aligned}
$$

The theorem can be easily proved by the definitions of the corresponding relations and matrices. From an indiscernibility or a discernibility matrix, we can easily obtain the indiscernibility and discernibility relations defined by any subset of attributes. On the other hand, from the family of all the strong indiscernibility or the strong discernibility relations defined by singleton subsets, we can obtain the indiscernibility or discernibility matrix.

Example 2 *The indiscernibility and discernibility matrices of the information Table 1 are illustrated in Tables 2 and 3. In the tables, for simplicity, we write a set of attributes, for example, $\{a, b, c\}$ as abc. Since both matrices are symmetric, we only list the elements in the upper right half.*

4 Attribute Reduction

An important application of the notions of indiscernibility/discernibility relations and matrices is the simplification of an information table. The problem of attribute reduction is the column-wise simplification of the table. Although the reduced table has less number of attributes, it preserves certain information of the original table.

Table 2
The indiscernibility matrix of the information Table 1.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1	At	$acef$	cd	$abcf$	$acef$	de	$acdef$	ef	ad
o_2		At	bc	$acdf$	At	be	$abcef$	$bdef$	ab
o_3			At	ce	bc	$abdf$	bcd	ab	$bdef$
o_4				At	$acdf$	$\emptyset$	acf	df	ae
o_5					At	be	$abcef$	$bdef$	ab
o_6						At	bde	$abce$	$bcdf$
o_7							At	bef	abd
o_8								At	bc
o_9									At

Table 3
The discernibility matrix of the information Table 1.

	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
o_1	$\emptyset$	bd	$abef$	de	bd	$abcf$	b	$abcd$	$bcef$
o_2		$\emptyset$	$adef$	be	$\emptyset$	$acdf$	d	ac	$cdef$
o_3			$\emptyset$	$abdf$	$adef$	ce	aef	$cdef$	ac
o_4				$\emptyset$	be	At	bde	$abce$	$bcdf$
o_5					$\emptyset$	$acdf$	d	ac	$cdef$
o_6						$\emptyset$	acf	df	ae
o_7							$\emptyset$	acd	cef
o_8								$\emptyset$	$adef$
o_9									$\emptyset$

4.1 Indiscernibility and discernibility reducts

A general definition of a reduct consists of two conditions. The first one ensures that a certain property is preserved by a set of attributes. Therefore, the attribute set as a whole is sufficient for preserving the property. The second condition ensures that the constructed attribute set is the minimum, i.e., each attribute of it is necessary for reserving the property.

Definition 7 *For an information table* $S = (U, At, \{V_a\}, \{I_a\})$*, an attribute* *set $P \subseteq At$ is a reduct if it meets the following two conditions:*

(1.) $\mathcal{R}(P) = \mathcal{R}(At)$;
(2.) For any proper subset $P' \subset P$, $\mathcal{R}(P') \neq \mathcal{R}(At)$,

where $\mathcal{R} \in \{\text{IND}, \text{DIS}\}$. *When* $\mathcal{R}$ *represents* IND, *the corresponding reducts are called indiscernibility reducts, and* $\mathcal{R} = \text{DIS}$ *induces discernibility reducts.*

The interpretation of these two reducts can be explained as follows:

- An indiscernibility reduct is a minimum attribute set that retains the strong indiscernibility relation defined by the entire attribute set. According to the condition (1.), an attribute set P might be an indiscernibility reduct if it satisfies the following condition:

  $$\forall (x, y) \in U \times U, (x, y) \in \text{IND}(At) \Leftrightarrow (x, y) \in \text{IND}(P).$$

  It means that if an object pair is strongly indiscernible regarding At, it is also strongly indiscernible regarding P, and vice versa. An attribute set P is not an indiscernibility reduct if it meets any of the following two conditions:
  $\exists (x, y) \in U \times U$ if $(x, y) \in \text{IND}(At)$ then $(x, y) \in \text{WDIS}(P)$ or
  if $(x, y) \in \text{WDIS}(At)$ then $(x, y) \in \text{IND}(P)$.
  It means that an object pair is strongly indiscernible regarding At but weakly discernible regarding P, or the object pair is weakly discernible regarding At but strongly indiscernible regarding P. According to the condition (2.), none subset $P' \subset P$ can retain the strong indiscernibility relation defined by the entire attribute set.
- A discernibility reduct is a minimum attribute set that retains the strong discernibility relation defined by the entire attribute set. According to the condition (1.), an attribute set P might be a discernibility reduct if it satisfies the following condition:

  $$\forall (x, y) \in U \times U, (x, y) \in \text{DIS}(At) \Leftrightarrow (x, y) \in \text{DIS}(P).$$

  It means that if an object pair is strongly discernible regarding At, it is also strongly discernible regarding P, and vice versa. An attribute set P is not a reduct if it meets any of the following two conditions:
  $\exists (x, y) \in U \times U$ if $(x, y) \in \text{DIS}(At)$ then $(x, y) \in \text{WIND}(P)$ or
  if $(x, y) \in \text{WIND}(At)$ then $(x, y) \in \text{DIS}(P)$.
  It means that an object pair is strongly discernible regarding At but weakly indiscernible regarding P, or the object pair is weakly indiscernible regarding At but strongly discernible regarding P. According to the condition (2.), none subset $P' \subset P$ can retain the strong discernibility relation defined by the entire attribute set.

Definition 8 *For an information table* $S = (U, At, \{V_a\}, \{I_a\})$, *an attribute* *set $P \subseteq At$ is an indiscernibility-and-discernibility reduct if it meets following three conditions:*

(1.) $\mathrm{IND}(P) = \mathrm{IND}(At)$*;*
(2.) $\mathrm{DIS}(P) = \mathrm{DIS}(At)$*;*
(3.) For any proper subset $P' \subset P$, either $\mathrm{IND}(P') \neq \mathrm{IND}(At)$ *or* $\mathrm{DIS}(P') \neq \mathrm{DIS}(At)$*.*

An indiscernibility-and-discernibility reduct is a minimum attribute set that keeps both the indiscernibility and the discernibility relations of the original table.

Obviously, the three reducts are defined by the strong relations. Alternatively, reducts can be defined by the weak relation counterparts and corresponding matrices. Theorem 2 derived from Theorem 1 establishes some connections between reducts and the properties of relations and matrices.

Theorem 2 *Let $S = (U, At, \{V_a\}, \{I_a\})$ be an information table. For a strong relation $\mathcal{R} \in \{\mathrm{IND}, \mathrm{DIS}\}$, its complement relation (in the weak form) and the other pair of relations are denoted as $W\mathcal{R}^c$, $\mathcal{R}^c$ and $W\mathcal{R}$, respectively. $\mathbf{m}_{\mathcal{R}}$ and $\mathbf{m}_{\mathcal{R}^c}$ are two complement matrices. The following conditions are equivalent: For any reduct P,*

(i.) $\mathcal{R}(P) = \mathcal{R}(At)$*;*
(ii.) $W\mathcal{R}^c(P) = W\mathcal{R}^c(At)$*;*
(iii.) *for all* $\mathbf{m}_{\mathcal{R}^c} \neq \emptyset$, $P \cap \mathbf{m}_{\mathcal{R}^c} \neq \emptyset$*;*
(iv.) *for all* $\mathbf{m}_{\mathcal{R}} \neq At$, $P \not\subseteq \mathbf{m}_{\mathcal{R}}$*.*

According to Theorem 2, Definition 7 can be redefined in the other three ways.

The family of all indiscernibility reducts of the information table S is denoted as $RED_{\mathrm{IND}}(S)$, and the family of all discernibility reducts as $RED_{\mathrm{DIS}}(S)$. The intersection of all indiscernibility reducts is called the IND core, and the intersection of all discernibility reducts is called the DIS core.

4.2 Indiscernibility and discernibility matrix functions

Skowron and Rauszer [21] define a *discernibility function* for a discernibility matrix $\mathbf{dm}$:

$$f_{\mathrm{DIS}}(\mathbf{dm}) = \bigwedge\{\bigvee \mathbf{dm}(x, y) \mid x, y \in U,\ \mathbf{dm}(x, y) \neq \emptyset\},$$

where $\bigvee \mathbf{dm}(x, y)$ represents the logical disjunction of all the attributes in an element $\mathbf{dm}(x, y)$, which means that x and y are discernible regarding any attribute in $\mathbf{dm}(x, y)$. A discernibility function is the conjunction of all the logical disjunction of matrix elements; it keeps all the weak discernibility relations in the given universe.

Similar to the discernibility function, we can define an *indiscernibility function* for an indiscernibility matrix **im**:

$$f_{\mathrm{IND}}(\mathbf{im}) = \bigwedge\{\bigvee \mathbf{im}(x, y) \mid x, y \in U,\ \mathbf{im}(x, y) \neq At\},$$

where $\bigvee \mathbf{im}(x, y)$ is the logical disjunction of all the attributes in an element $\mathbf{im}(x, y)$, which means that x and y are indiscernible regarding any attribute in $\mathbf{im}(x, y)$. An indiscernibility function is the conjunction of all the logical disjunction of matrix elements; it keeps all the weak indiscernibility relations in the given universe.

Theorem 3 *The problem for constructing the family of reducts $RED_{\mathcal{R}}(S)$, where $\mathcal{R} \in \{\mathrm{IND}, \mathrm{DIS}\}$, is equivalent to the problem of transforming any conjunctive form of a matrix function $f_{\mathcal{R}}$ to a reduced disjunctive form.*

Proof. After transforming a matrix function $f_{\mathcal{R}}$ to a disjunctive form by applying multiplication and absorption laws whenever possible, we get the reduced disjunctive form of $f_{\mathcal{R}}$. A disjunct of a matrix function $f_{\mathcal{R}}$ in a reduced disjunctive form is called a *prime implicant* [21], which possesses the following properties:

(i.) for each prime implicant, there exists a reduct, such that the conjunction of all the attributes forming that reduct is equivalent to the prime implicant, and
(ii.) for each reduct, there exists a prime implicant, which is obtained by taking the conjunction of all the attributes forming that reduct. □

Skowron and Rauszer explicitly state the strong connection between the notions of a reduct in an information table and a prime implicant of the matrix function [21]. Here, we put it more clearly, an indiscernibility reduct and a disjunct of $f_{\mathrm{DIS}}(\mathbf{dm})$ have a strong connection, namely, we have the following equivalence:

$$P \in RED_{\mathrm{IND}}(S) \text{ iff } \bigwedge P \text{ is a prime implicant of } f_{\mathrm{DIS}}(\mathbf{dm}).$$

Similarly, we can conclude that a discernibility reduct and a disjunct of $f_{\mathrm{IND}}(\mathbf{im})$ have a strong connection, namely:

$$P \in RED_{\mathrm{DIS}}(S) \text{ iff } \bigwedge P \text{ is a prime implicant of } f_{\mathrm{IND}}(\mathbf{im}).$$

Example 3 *Based on the discernibility matrix in Table 3, we can construct* *the following discernibility function:*

$$
\begin{aligned}
f_{\mathrm{DIS}}\ (\mathbf{dm}) = {} & b \wedge d \wedge (a \vee c) \wedge (a \vee e) \wedge (b \vee d) \wedge (b \vee e) \wedge (c \vee e) \wedge (d \vee e) \wedge \\
& (d \vee f) \wedge (a \vee c \vee d) \wedge (a \vee c \vee f) \wedge (a \vee e \vee f) \wedge (b \vee d \vee e) \wedge (c \vee e \vee f) \wedge \\
& (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee e) \wedge (a \vee b \vee c \vee f) \wedge (a \vee b \vee d \vee f) \wedge \\
& (a \vee b \vee e \vee f) \wedge (a \vee c \vee d \vee f) \wedge (a \vee d \vee e \vee f) \wedge (b \vee c \vee d \vee f) \wedge \\
& (b \vee c \vee e \vee f) \wedge (c \vee d \vee e \vee f) \wedge (a \vee b \vee c \vee d \vee e \vee f) \\
= {} & b \wedge d \wedge (a \vee c) \wedge (a \vee e) \wedge (c \vee e) \\
= {} & (a \wedge b \wedge c \wedge d) \vee (a \wedge b \wedge d \wedge e) \vee (b \wedge c \wedge d \wedge e).
\end{aligned}
$$

Based on the indiscernibility matrix in Table 2, we can construct the following indiscernibility function:

$$
\begin{aligned}
f_{\mathrm{IND}}\ (\mathbf{im}) = {} & (a \vee b) \wedge (a \vee d) \wedge (a \vee e) \wedge (b \vee c) \wedge (b \vee e) \wedge (c \vee d) \wedge (c \vee e) \wedge \\
& (d \vee e) \wedge (d \vee f) \wedge (e \vee f) \wedge (a \vee b \vee d) \wedge (a \vee c \vee f) \wedge (b \vee c \vee d) \wedge \\
& (b \vee d \vee e) \wedge (b \vee e \vee f) \wedge (a \vee b \vee c \vee e) \wedge (a \vee b \vee c \vee f) \wedge (a \vee b \vee d \vee f) \wedge \\
& (a \vee c \vee d \vee f) \wedge (a \vee c \vee e \vee f) \wedge (b \vee c \vee d \vee f) \wedge (b \vee d \vee e \vee f) \wedge \\
& (a \vee b \vee c \vee e \vee f) \wedge (a \vee c \vee d \vee e \vee f) \\
= {} & (a \vee b) \wedge (a \vee d) \wedge (a \vee e) \wedge (b \vee c) \wedge (b \vee e) \wedge (c \vee d) \wedge (c \vee e) \wedge (d \vee e) \wedge \\
& (d \vee f) \wedge (e \vee f) \wedge (a \vee c \vee f) \\
= {} & (a \wedge b \wedge d \wedge e) \vee (a \wedge c \wedge d \wedge e) \vee (a \wedge c \wedge e \wedge f) \vee (b \wedge c \wedge d \wedge e) \vee \\
& (b \wedge d \wedge e \wedge f) \vee (a \wedge b \wedge c \wedge d \wedge f).
\end{aligned}
$$

It means that we obtain three indiscernibility reducts ($\{a, b, c, d\}$, $\{a, b, d, e\}$ and $\{b, c, d, e\}$) and six discernibility reduces ($\{a, b, d, e\}$, $\{a, c, d, e\}$, $\{a, c, e, f\}$, $\{b, c, d, e\}$, $\{b, d, e, f\}$, and $\{a, b, c, d, f\}$) of the information Table 1.

According to Definition 8 of an indiscernibility-and-discernibility reduct, we can define an *indiscernibility-and-discernibility function* as follows:

$$
f_{\mathrm{IND\text{–}DIS}}(\mathbf{im\text{–}dm}) = f_{\mathrm{IND}} \bigwedge f_{\mathrm{DIS}}.
$$

The problem for constructing the family of indiscernibility-and-discernibility reducts $RED_{\mathrm{IND\text{–}DIS}}(S)$ is equivalent to the problem of transforming the conjunctive form of two matrix functions f_{IND} and f_{DIS} to a reduced disjunctive form.

Example 4 *Based on the results of our previous example,*

$$
\begin{aligned}
f_{\text{IND–DIS}}\ (\mathbf{im–dm}) &= f_{\text{IND}} \bigwedge f_{\text{DIS}} \\
&= ((a \wedge b \wedge c \wedge d) \vee (a \wedge b \wedge d \wedge e) \vee (b \wedge c \wedge d \wedge e)) \bigwedge \\
&\quad ((a \wedge b \wedge d \wedge e) \vee (a \wedge c \wedge d \wedge e) \vee (a \wedge c \wedge e \wedge f) \vee (b \wedge c \wedge d \wedge e) \\
&\quad \vee (b \wedge d \wedge e \wedge f) \vee (a \wedge b \wedge c \wedge d \wedge f)) \\
&= (a \wedge b \wedge d \wedge e) \vee (b \wedge c \wedge d \wedge e) \vee (a \vee b \wedge c \wedge d \wedge f).
\end{aligned}
$$

The result shows that we can obtain three indiscernibility-and-discernibility reducts, i.e., $\{a, b, d, e\}$ $\{b, c, d, e\}$ *and* $\{a, b, c, d, f\}$ *of Table 1.*

Although a matrix function expressed in a reduced disjunctive normal form is ideal for finding all reducts, it can be very complex while the amount of matrix elements is huge. In this case, a variety of heuristics need to be applied to find one reduct, a group of reducts, or a preferred reduct. Many promising search strategies and search heuristics have been proposed and tested for constructing indiscernibility reducts. They equally can be applied to construct discernibility reducts and indiscernibility-and-discernibility reducts.

5 Relative Reducts and Relative Reduct Construction

Without losing the generality, we can split the attribute set At into two parts, i.e., $At = C \cup D$, where C is a set of conditional attributes, and D is a set of decision attributes. Such an information table is also called a decision table for decision making purposes. For simplicity, in the rest of the paper, we only treat D as a singleton set, and the only attribute in the set is also called D.

5.1 Relative relations and matrices

Given a subset of conditional attributes $A \subseteq C$, two decision-relative relations, briefly, relative relations or D-relative relations, between objects can be distinguished.

Definition 9 *Given a subset of conditional attributes $A \subseteq C$, the four relative relations on U are defined as:*

$$
\begin{aligned}
\text{IND}_D(A) &= \{(x, y) \in U \times U \mid \forall a \in A, I_a(x) = I_a(y) \text{ and } I_D(x) = I_D(y)\}, \\
\text{WIND}_D(A) &= \{(x, y) \in U \times U \mid \exists a \in A, I_a(x) = I_a(y) \text{ and } I_D(x) = I_D(y)\}, \\
\text{DIS}_D(A) &= \{(x, y) \in U \times U \mid \forall a \in A, I_a(x) \neq I_a(y) \text{ and } I_D(x) \neq I_D(y)\}, \\
\text{WDIS}_D(A) &= \{(x, y) \in U \times U \mid \exists a \in A, I_a(x) \neq I_a(y) \text{ and } I_D(x) \neq I_D(y)\}.
\end{aligned}
$$

The relation-counterpart matrices can be constructed. Each cell of a relative discernibility matrix stores those conditional attributes, on which the corresponding two objects of the universe have distinct values and different classes [21].

Definition 10 *Given a decision table S, its relative discernibility matrix* $\mathbf{dm}_D$ *is a* $|U| \times |U|$ *matrix. For all* $(x, y) \in U \times U$ *each element* $\mathbf{dm}_D(x, y)$ *is defined as:*

$$\mathbf{dm}_D(x,y) = \begin{cases} \{a \in C \mid I_a(x) \neq I_a(y)\}, & if\ I_D(x) \neq I_D(y), \\ \emptyset, & otherwise. \end{cases}$$

In contract to a relative discernibility matrix, a relative indiscernibility matrix can be similarly defined. Each element of a relative indiscernibility matrix stores the conditional attributes, on which the corresponding two objects have the same values and belong to the same class.

Definition 11 *Given a decision table S, its relative indiscernibility matrix* $\mathbf{im}_D$ *is a* $|U| \times |U|$ *matrix. For all* $(x, y) \in U \times U$ *each element* $\mathbf{im}_D(x, y)$ *is defined as:*

$$\mathbf{im}_D(x,y) = \begin{cases} \{a \in C \mid I_a(x) = I_a(y)\}, & if\ I_D(x) = I_D(y), \\ \emptyset, & otherwise. \end{cases}$$

Both of the relative indiscernibility and discernibility matrices are symmetric, i.e., $\mathbf{im}_D(x, x) = At$ and $\mathbf{dm}_D(x, x) = \emptyset$. To note, for a decision table, there are one and only one relative discernibility matrix $\mathbf{dm}_D$, but $|V_D|$ relative indiscernibility matrices regarding each value of D.

Example 5 *For the further illustration, we provide a sample decision Table 4 by adding a decision attribute D into the former information Table 1. The relative indiscernibility and discernibility matrices of the decision Table 4 are illustrated in Tables 5 and 6.*

5.2 Relative reducts and relative reduction

We can define two different relative reducts associated with the complementary relations and matrices.

Definition 12 *For a decision table* $S = (U, At = C \cup \{D\}, \{V_a\}, \{I_a\})$, *an*

Table 4
A decision table.

U	C						D
	a	b	c	d	e	f	
o_1	1	1	1	1	1	1	+
o_2	1	0	1	0	1	1	+
o_3	0	0	1	1	0	0	+
o_4	1	1	1	0	0	1	-
o_5	1	0	1	0	1	1	-
o_6	0	0	0	1	1	0	-
o_7	1	0	1	1	1	1	-
o_8	0	0	0	0	1	1	-
o_9	1	0	0	1	0	0	-

Table 5
Two relative indiscernibility matrices $\mathbf{im}_+$ and $\mathbf{im}_-$ of the decision Table 4.

	o_1	o_2	o_3
o_1	At	$acef$	cd
o_2		At	bc
o_3			At

	o_4	o_5	o_6	o_7	o_8	o_9
o_4	At	$acdf$	$\emptyset$	acf	df	ae
o_5		At	be	$abcef$	$bdef$	ab
o_6			At	bde	$abce$	$bcdf$
o_7				At	bef	abd
o_8					At	bc
o_9						At

Table 6
The relative discernibility matrix $\mathbf{dm}_D$ of the decision Table 4.

	o_4	o_5	o_6	o_7	o_8	o_9
o_1	de	bd	$abcf$	b	$abcd$	$bcef$
o_2	be	$\emptyset$	$acdf$	d	ac	$cdef$
o_3	$abdf$	$adef$	ce	aef	$cdef$	ac

attribute set $P \subseteq C$ is a D-relative reduct if it meets following two conditions:

(R1.) $\mathcal{R}_D(P) = \mathcal{R}_D(C)$;
(R2.) For any proper subset $P' \subset P$, $\mathcal{R}_D(P') \neq \mathcal{R}_D(C)$,

where $\mathcal{R} \in \{\text{IND}, \text{DIS}\}$. The corresponding reducts are called relative indiscernibility reducts and relative discernibility reducts, respectively.

Similar to Theorem 2, Definition 12 can be redefined in other three ways. All the methods for reduct construction can be used for relative reduct construction.

Example 6 *According to the relative indiscernibility matrices* $\mathbf{im}_+$ *and* $\mathbf{im}_-$ *shown in Table 5, we can obtain*

$$f_{\text{IND}_D}(\mathbf{im}_+) = c \vee (a \wedge b \wedge d) \vee (b \wedge d \wedge e) \vee (b \wedge d \wedge f).$$

Each disjunct indicates a positive-relative discernibility reduct, and,

$$f_{\text{IND}_D}(\mathbf{im}_-) = (a \wedge b \wedge d) \vee (a \wedge c \wedge d \wedge e) \vee (a \wedge c \wedge e \wedge f) \vee (b \wedge c \wedge d).$$

Each disjunct indicates a negative-relative discernibility reduct.

According to the relative discernibility matrix shown in Table 6, we can obtain

$$f_{\text{DIS}_D}(\mathbf{dm}_D) = (a \wedge b \wedge c \wedge d) \vee (a \wedge b \wedge d \wedge e) \vee (b \wedge c \wedge d \wedge e) \vee (b \wedge c \wedge d \wedge f).$$

Each disjunct indicates a relative indiscernibility reduct.

It needs to be noted that Susmaga's terminologies for reducts and constructs are closely related to the ones used in this paper. [27,28]. Susmaga uses the standard definition of an indiscernibility relation. The complement relation to an indiscernibility relation is called a discernibility relation. A similarity relation defined by a set of attributes indicates all the object pairs that are indiscernible on at least one attribute of the set. According to Susmaga, a reduct is a minimum set of attributes that retains the discernibility relation. Constructs, defined in a similar way, represent a notion that is a kind of generalization of the reduct. That is, a construct retains both the discernibility and similarity relations of the whole set of attributes. Table 7 summaries Susmaga's terminologies and the ones used in this paper. Susmaga's definition of reducts is equivalent to the definition of our indiscernibility reducts, and his definition of conducts is actually the definition of our indiscernibility-and-discernibility reducts. We extend Susmaga's research by adding discernibility reducts into the discussion. We do not want to use construct, or any other new words, to name a different type of reducts. The main differences among different types of reducts reside in the properties they retain. By our naming convention,

Susmaga's	Ours	Meanings
IND(A)	IND(A)	$\{(x, y) \in U \times U \mid \forall a \in A, I_a(x) = I_a(y)\}$
SIM(A)	WIND(A)	$\{(x, y) \in U \times U \mid \exists a \in A, I_a(x) = I_a(y)\}$
	DIS(A)	$\{(x, y) \in U \times U \mid \forall a \in A, I_a(x) \neq I_a(y)\}$
DIS(A)	WDIS(A)	$\{(x, y) \in U \times U \mid \exists a \in AI_a(x) \neq I_a(y)\}$
reducts	IND reducts	retain the IND and WDIS relations
	DIS reducts	retain the DIS and WIND relations
constructs	IND−DIS reducts	retain the strong and weak IND/DIS relations

Table 7
Comparing Susmaga's terminologies to the ones used in this paper

the retaining properties of indiscernibility reducts, discernibility reducts and indiscernibility-and-discernibility reducts are clearly stated as their names.

6 Experiment Evaluations

To demonstrate the usefulness of the complementary relations and reducts, we use a real-life dataset for letter recognition. There are two different tasks for English letter recognition. On one hand, we want to distinguish letters written by different people. To deal with this task, for a particular letter, we need to group together the letters written in the same style, and differentiate the groups of letters. On the other hand, in some situations, a letter written by the same person may diverse in a certain degree. We want to identify the letters written by the same person by ignoring certain differences. That is, we want to summarize the characteristics of one's writing style, and recognize the letter in order to identify the person's handwriting correctly. These two tasks are especially useful for signature identification. We can use reducts to represent the rules and simplify the rules.

The objective for constructing the indiscernibility reducts of the dataset is to retain the strong indiscernibility relation and the weak discernibility relation. Suppose the letters are written by different people. A strong indiscernibility relation groups the letters that are the same regarding the describing attributes. An indiscernibility reduct is a minimum attribute set that can distinguish these groups. For any subset of the indiscernibility reduct, more object pairs, weakly discernible regarding the entire attribute set, are incorrectly grouped together.

The objective for constructing the discernibility reducts of the dataset is to retain the strong discernibility relation and the weak indiscernibility relation.

Suppose some of the letters are written by the same person. A strong discernibility relation lists all the distinct object pairs regarding to their values on the describing attributes. A discernibility reduct is a minimum attribute set that retains the distinct object pairs. For any subset of the discernibility reduct, more object pairs, weakly indiscernible regarding the entire attribute set, are incorrectly identified as distinct pairs.

The dataset used is called *letter-recognition* from the UCI Machine Learning Repository. It contains 20,000 unique data records, each represents a handwritten capital letter in the English alphabet. Each letter is identified as a group of black-and-white pixels in a rectangular box, and is conveniently described by one categorical attribute and sixteen primitive numerical attributes that represent the statistical values and edge counts of the pixel box. Each numerical attribute has sixteen possible values from 0 to 15. The meaning of attributes are listed below:

1.	capital letter	(26 values from A to Z)
2.	horizontal position of box: x	(integer)
3.	vertical position of box: y	(integer)
4.	width of box	(integer)
5.	height of box	(integer)
6.	total number of on-pixels	(integer)
7.	x of on-pixels in box	(integer)
8.	y of on-pixels in box	(integer)
9.	x variance	(integer)
10.	y variance	(integer)
11.	x y correlation	(integer)
12.	$x * x * y$	(integer)
13.	$x * y * y$	(integer)
14.	edge count left to right	(integer)
15.	correlation of x-edge with y	(integer)
16.	edge count bottom to top	(integer)
17.	correlation of y-edge with x	(integer)

We partition the 20,000 objects according to the letters. The class distribution is almost equalized. We randomly pick two sub-tables containing letter A (789 records) and the letter S (748 records) for our evaluation. For each sub-table, the letters are described by the sixteen numerical attributes. We want to construct the indiscernibility reducts and discernibility reducts of these two tables.

The method applied for this experiment is a greedy algorithm. For sixteen attributes, we have $2^{16} - 1 = 65,535$ subsets. We relax the strong indiscernibility and the strong discernibility relations, such that if $(x, y) \in \mathrm{ind}_{90\%}(A)$ then x and y are considered indiscernible. If $(x, y) \in \mathrm{dis}_{90\%}(A)$ then x and y

	# of IND-reducts	# of DIS-reducts
Length = 3	17	0
Length = 4	33	0
Length = 5	19	0
Length = 6	9	0
Length = 7	1	0
Length = 8	0	1
Length = 9	0	5
Total	79	6

Table 8
Comparing indiscernibility/discernibility reducts of the table containing the letter A

	# of IND-reducts	# of DIS-reducts
Length = 3	1	0
Length = 4	45	0
Length = 5	46	0
Length = 10	0	5
Length = 11	0	10
Length = 12	0	1
Total	92	16

Table 9
Comparing indiscernibility/discernibility reducts of the table containing the letter S

are considered discernible. The counts of indiscernibility/discernibility reducts of two tables are listed in Tables 8 and 9, respectively.

From the results, both the indiscernibility and discernibility reducts can simply the table and rules. We can tell that we usually have more indiscernibility reducts than discernibility reducts, and the lengths of indiscernibility reducts are shorter than discernibility reducts. It shows that it is easier to distinguish letters written in different styles, and harder to identify letters written in the same style.

7 Conclusion

Data analysis approaches are mainly based on two different views, i.e., to explore either the similarities or the differences of objects. To cope with these two views, four relations can be observed. They are the strong and weak indiscernibility relations and the strong and weak discernibility relations. Alternatively, two complementary matrices, an indiscernibility matrix and a discernibility matrix, can be applied for data analysis.

Based on the two views, three different types of reducts can be constructed. They are indiscernibility reducts, discernibility reducts and indiscernibility-and-discernibility reducts. An indiscernibility reduct is a minimum attribute set that retains the strong indiscernibility relation and the weak discernibility relation defined by the entire attribute set. A discernibility reduct is a minimum attribute set that retains the strong discernibility relation and the weak indiscernibility relation defined by the entire attribute set. Combining these two features, an indiscernibility-and-discernibility reduct retains all the four relations defined by the entire attribute set. Although the indiscernibility reducts have been extensively studied by the rough set society, the other two types of reducts need further exploration.

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References

[1] Bazan, J.G., Nguyen, H.S., Nguyen, S.H., Synak, P. and Wróblewski, J., Rough set algorithms in classification problems, in: Polkowski, L., Tsumoto, S. and Lin, T.Y. (Eds.) *Rough Set Methods and Applications*, Physica Verlag, Heidelberg, 49-88, 2000.

[2] Chen, D., Wang, C. and Hu, Q., A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets, *Information Sciences*, 2007, in press.

[3] Demri, S. and Orlowska, E., Logical analysis of indiscernibility, in: Orlowska, E. (Ed.), *Incomplete Information: Rough Set Analysis*, Physica-Verlag, Heidelberg, 347-380, 1998.

[4] Grzymala–Busse, J.W., LERS: a system for learning from examples based on rough sets, in: Słowiński, R. (Ed.) *Intelligent Decision Support: Handbook of Advances and Applications of the Rough Sets Theory*, Kluwer, Dordrecht, 3-18, 1992.

[5] Grzymala-Busse, J.W., Characteristic relations for incomplete data: a generalization of the indiscernibility Relation, *Transactions of Rough Sets*, 58-68, 2005.

[6] Hadjumichael, M. and Wong, S.K.M., Fuzzy representation in rough set approximations, *International Workshop on Rough Sets and Knowledge Discovery in Databases*, 349-356, 1993.

[7] Kokar, M.M., On similarity methods in machine learning, *Proceedings of the International Workshop on Similarity Methods*, 15-25, 2000.

[8] Mi, J.S., Wu, W.Z. and Zhang, W.X., Approaches to knowledge reduction based on variable precision rough set model, *Information Sciences*, 159, 255-272, 2004.

[9] Nguyen, S.H., Regularity analysis and its applications in data mining, in: Polkowski, L., Tsumoto, S. and Lin, T.Y. (Eds.) *Rough Set Methods and Applications*, Physica Verlag, Heidelberg, 289-378, 2000.

[10] Novotný, M. and Pawlak, Z., On a representation of rough sets by means of information systems, *Fundamenta Informaticae*, 6, 286-296, 1983.

[11] Orlowska, E., Semantics analysis of inductive reasoning, *Theoretical Computer Science*, 43, 81-89, 1986.

[12] Orlowska, E., Introduction: what you always wanted to know about rough sets, in: Orlowska, E. (Ed.) *Incomplete Information: Rough Set Analysis*, Physica Verlag, Heidelberg, New York, 1-20, 1998.

[13] Pawlak, Z., *Rough Sets: Theoretical Aspects of Reasoning About Data*, Kluwer Academic Publishers, Dordrecht, MA, 1991.

[14] Pawlak, Z., Granularity of knowledge, indiscernibility and rough sets, *Proceedings of the IEEE International Conference on Fuzzy Systems*, 106-110, 1998.

[15] Pawlak, Z., Conflicts and negotiations, *Proceedings of Rough Sets and Knowledge Technology*, 12-27, 2006.

[16] Polkowski, L. and Skowron, A., Rough mereology, *Proceedings of ISMIS*, 85-94, 1994.

[17] Polkowski, L., Skowron, A. and Zytkow, J., Tolerance based rough sets, in: Lin, T.Y. and Wildberger, A.M. (Eds.) *Soft Computing: Rough Sets, Fuzzy Logic, Neural Networks, Uncertainty Management*, 55-58, 1995.

[18] Qiu, G.F., Zhang, W.X. and Wu, W.Z., Characterizations of attributes in generalized approximation representation spaces, *Proceedings of the 10th International Conference on RSFDGrC*, 84-93, 2005.

[19] Quafafou, M., α-RST: A generalization of rough set theory, *Information Sciences*, 124, 301-316, 2000.

[20] Skowron, A. and Polkowski, L., Rough mereological foundations for design, analysis, synthesis, and control in distributed systems, *Information Sciences*, 104, 129-156, 1998.

[21] Skowron, A. and Rauszer, C., The discernibility matrices and functions in information systems, R. Slowiński (Ed.) *Intelligent Decision Support, Handbook of Applications and Advances of the Rough Sets Theory*, Kluwer Academic Publishers, Dordrecht, 1992.

[22] Skowron A. and Stepaniuk J., Decision rules based on discernibility matrices and decision matrices, *Proceedings of the International Workshop on Rough Sets and Soft Computing (RSSC'94)*, 156-163, 1994.

[23] Slezak, D., Various approaches to reasoning with frequency based decision reducts: a survey, in: Polkowski, L., Tsumoto, S., Lin, T.Y. (Eds.) *Rough set methods and applications*. Physica-verlag, Heidelberg, 235-285, 2000.

[24] Slowinski, R. and Vanderpooten, D., A generalized definition of rough approximation based on similarity, *IEEE Transaction on Data and Knowledge Engineering*, 12(2), 331-336, 2000.

[25] Slowinski, R. and Stefanowski, J., Handling various types of uncertainty in the rough set approach, *Proceedings of the International Workshop on Rough Sets and Knowledge Discovery*, 366-376.

[26] Stefanowski J. and Tsoukias A., Valued tolerance and decision rules, *Proceedings of Rough Sets and Current Trends in Computing*, 170-181, 2000.

[27] Susmaga, R., Tree-like parallelization of reduct and construct computation, *Proceedings of Rough Sets and Current Trends in Computing*, 455-464, 2004.

[28] Susmaga, R., Reducts and constructs in attribute reduction, *Fundamenta Informaticae*, 61(2), 159-181, 2004.

[29] Swiniarski, R.W., Rough sets methods in feature reduction and classification, *International Journal of Applied Mathematics and Computer Science*, 11(3), 565-582, 2001.

[30] Wang, G.Y., Rough reduction in algebra view and information view, *International Journal of Intelligent Systems*, 18(6), 679-688, 2003.

[31] Wang, J. and Wang, J., Reduction algorithms based on discernibility matrix: the ordered attributes method, *Journal of Computer Science and Technology*, 16, 489-504, 2001.

[32] Wong, S.K.M. and Ziarko, W., On optimal decision rules in decision tables, *Bulletin of the Polish Academy of Sciences and Mathematics*, 693-696, 1985.

[33] Yao, Y.Y., Relational interpretations of neighborhood operators and rough set approximation operators, *Information Sciences*, 111(1-4), 239-259, 1998.

[34] Yao, Y.Y. and Lin, T. Y., Generalization of rough sets using modal logic, *Intelligent Automation and Soft Computing*, 2, 103-120, 1996.

[35] Yao, Y.Y. and Wong, S.K.M., Generalization of rough sets using relaitonships between attribute values, *Proceedings of the Joint Conference of Information Sciences*, 30-33, 1995.

[36] Zhao, K. and Wang, J., A reduction algorithm meeting users' requirements, *Journal of Computer Science and Technology*, 17, 578-593, 2002.

[37] Zhu, W. and Wang, F.Y., Reduction and axiomization of covering generalized rough sets, *Information Sciences*, 152, 217-230, 2003.

[38] Ziarko, W., Rough set approaches for discovering rules and attribute dependencies, in: Klösgen, W., Żytkow, J.M. (Eds.) *Handbook of Data Mining and Knowledge Discovery*, Oxford, 328-339, 2000.