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Constructive and Algebraic Methods of the Theory of Rough Sets

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This paper reviews and compares constructive and algebraic approaches in the study of rough sets. In the constructive approach, one starts from a binary relation and defines a pair of lower and upper approximation operators using the binary relation. Different classes of rough set algebras are obtained from different types of binary relations. In the algebraic approach, one defines a pair of dual approximation operators and states axioms that must be satisfied by the operators. Various classes of rough set algebras are characterized by different sets of axioms. Axioms of approximation operators guarantee the existence of certain types of binary relations producing the same operators.

1 INTRODUCTION

The theory of rough sets deals with the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations [31]. There are at least two methods for the development of this theory, the constructive and algebraic (axiomatic) approaches. In constructive methods, lower and upper approximations are not primitive notions. They are constructed from other concepts, such as binary relations on a universe [1,31,46], partitions and coverings of a universe [34,35,49], and partially ordered sets, lattice, Boolean algebras and their subalgebras [2,8,11,24]. On the other hand, by treating lower and upper approximations as primitive notions, algebraic (axiomatic) methods focus on algebraic systems for the theory of rough sets. A set of axioms is used to characterize approximation operators that are the same as the ones produced using constructive methods [5,6,18,42,44].

Studies of rough set theory may be divided into two classes, representing the set-oriented and operator-oriented views of the theory [44]. They produce deviation and extension of classical set algebra [45]. With set-oriented view, the pair of lower and upper approximations is used to define an equivalence relation on the power set of the universe. Two subsets are said to be equivalent if they have the same lower and upper approximations. An equivalence class of this equivalence relation is referred to as a rough set [31]. Under this interpretations, one cannot use the standard set-theoretic operators for rough set operators. Instead, rough set operators are interpreted by using operations in mathematical systems such as Stone algebra, Nelson algebra (quasi-pseudo Boolean algebra), and Heyting algebra (pseudo-Boolean algebra) [6,7,28,29,36,39]. The operator-oriented view interprets lower and upper approximations as a pair of unary set-theoretic operators on the power set of the universe [46]. This produces an augment of set algebra with added operators.

The majority of studies on rough sets have been concentrated on constructive approaches. Less effort has been made for axiomatic approaches, and particularly for the operator-oriented view. Based on results from modal logic [3] and its relationship to rough sets [13,22,25,27], Yao and Lin [46] analyzed various rough set models. It provides a systematic study on constructive methods using binary relations. Some important axiomatic studies on operator-oriented view are summaried below. Zakowski [48] studied a set of axioms on approximation operators. A problem with this study is that an equivalence relation is explicitly used in the definition of lower and upper approximations. Comer [5,6] investigated axioms on approximation operators in relation to cylindric algebras. The investigation is made within the context of Pawlak information systems [30]. Lin and Liu [18] suggested six axioms on a pair of abstract operators on the power set of universe in the framework of topological spaces. Under these axioms, there exists an equivalent relation such that the lower and upper approximations are the same as the abstract operators. The similar result was stated earlier by Wiweger [40]. All those studies are restricted to Pawlak rough set algebra defined by equivalence relations. Wybraniec-Skardowska [42,43] examined many axioms on various classes of approximation operators. Different constructive methods were suggested to produce such approximation operators. Yao [44] extended axiomatic approach to rough set algebras constructed from arbitrary binary relations.

Based on the above studies, this paper reviews, examines, and compares constructive and algebraic methods of the theory of rough sets. For clarity, our discussions will be restricted to binary relation based construction methods and operator-oriented view of the theory. Since constructive methods have been investigated in great detail in [46], the present study will foucs mainly on algebraic methods. Algebraically speaking, both rough set theory and propositional modal logic can be related to Boolean algebra with added operators [10,17,37,38]. The isomorphisms between Boolean algebra with added operators, rough set theory, and propositional modal logic guarantees that every theorem in any one of these theories has a counterpart in the other theory. These isomorphisms enable us to cover all these theories by developing one of them [13]. In parallel to the axiomatic study of modal logic, we develops an axiomatic approach for the theory of rough sets. The majority of the paper is devoted to the analysis of axioms of approximation operators, and the examination of conditions under which different rough set algebras are defined. The necessary and sufficient conditions for the equivalence of several commonly used definitions of approximation operators are studied.

2 TWO METHODS OF THE THEORY OF ROUGH SETS

For completeness, the constructive approach of the theory of rough sets based on binary relations is first reviewed [46]. The algebraic approach is then analyzed. Finally, the connections between the two approaches are studied.

2.1 CONSTRUCTIVE METHOD

Let U denote a finite and nonempty set called the universe. Suppose $R \subseteq U \times U$ is an equivalence relation on U, i.e., R is reflexive, symmetric, and transitive. The equivalence relation R partitions the set U into disjoint subsets. It is a quotient set of the universe and is denoted by U/R. Elements in the same equivalence class are said to be indistinguishable. Equivalent classes of R are called elementary sets (atoms). Every union of elementary sets is called a definable (composed) set [31,32]. The empty set is considered to be a definable set. Given an arbitrary set $X \subseteq U$, one can characterize X by a pair of lower and upper approximations. The lower approximation $\underline{apr}_R X$ is the greatest definable set contained in X, and the upper approximation $\overline{apr}_R X$ is the least definable set containing X. They can be computed by two equivalent formulas:

$$
\begin{aligned}
&\text{(C1)} \quad && \underline{apr}_R X = \{x \mid [x]_R \subseteq X\}, \\
& && \overline{apr}_R X = \{x \mid [x]_R \cap X \neq \emptyset\}; \\
&\text{(C2)} \quad && \underline{apr}_R X = \bigcup \{[x]_R \mid [x]_R \subseteq X\}, \\
& && \overline{apr}_R X = \bigcup \{[x]_R \mid [x]_R \cap X \neq \emptyset\},
\end{aligned}
$$

where

$$[x]_R = \{y \mid xRy\}, \tag{1}$$

is the equivalence class containing x. Lower and upper approximations can be regarded as defining two operators on the power set of U. They are dual operators in the sense one can be defined by the other using formulas:

$$\begin{array}{ll} \text{(L0)} & \underline{apr}_R X = \sim \overline{apr}_R \sim X, \\ \text{(H0)} & \overline{apr}_R X = \sim \underline{apr}_R \sim X. \end{array}$$

The upper approximation of a singleton subset $\{x\}$ is the equivalent class containing the element x, namely,

$$\overline{apr}_R\{x\} = [x]_R. \tag{2}$$

By combining it with (C1) and (C2), the lower and upper approximation operator can be expressed in terms of upper approximations of singleton subsets:

$$\begin{array}{ll} \text{(C3)} & \underline{apr}_R X = \{x \mid \overline{apr}_R\{x\} \subseteq X\}, \\ & \overline{apr}_R X = \{x \mid \overline{apr}_R\{x\} \cap X \neq \emptyset\}; \\ \text{(C4)} & \underline{apr}_R X = \bigcup\{\overline{apr}_R\{x\} \mid \overline{apr}_R\{x\} \subseteq X\}, \\ & \overline{apr}_R X = \bigcup\{\overline{apr}_R\{x\} \mid \overline{apr}_R\{x\} \cap X \neq \emptyset\}. \end{array}$$

Thus, one can define approximation operators on all singleton subsets and extend them to all other subsets.

The two distinct but equivalent definitions (C1) and (C2) stem from the fact that R is a equivalence relation. According to (C2), the lower approximation $\underline{apr}_R X$ is the union of all the equivalence classes which are subsets of X, and the upper approximation $\overline{apr}_R X$ is the union of all the equivalence classes which have a nonempty intersection with X. By (C1), an element belongs to the lower approximation if *all* its equivalent elements belong to X, and an element belongs to the upper approximation is at least *one* of its equivalent elements belongs to X. They can be reexpressed as follows:

$$\begin{array}{ll} \text{(C1)} & \underline{apr}_R X = \{x \mid \forall y[xRy \Longrightarrow y \in X]\}, \\ & \overline{apr}_R X = \{x \mid \exists y[xRy, y \in X]\}. \end{array}$$

Under this interpretation, the notions of lower and upper approximations are closely related to the necessity and possibility operators in modal logic [46]. An element equivalent to a member in the lower approximation *necessarily* belongs to X, while an element equivalent to a member in the upper approximation *possibly* belongs to X. Elements of the lower approximation are referred to as *strong* members, while elements of the upper approximation are *weak* members [31].

The Pawlak rough sets can be easily extended by considering other types of binary relations [42,46]. Let $R \subseteq U \times U$ be a binary relation on U without any additional constraints. For two elements $x, y \in U$, if xRy, we say that y is R-related to x, x is a predecessor of y, and y is a successor of x. A binary relation can be equivalently defined by a mapping from U to the power set 2^U, namely,

$$R_s(x) = \{y \mid xRy\}. \tag{3}$$

The set $R_s(x)$ of R-related elements of x may be interpreted as a successor neighborhood of x [46]. When the relation is an equivalence relation, $R_s(x)$ is the equivalence class containing x. In general, by substituting $[x]_R$ with $R_s(x)$, one may approximate a subset $X \subseteq U$ by a pair of subsets of U with respect to any binary relation.

Definition 1 *Suppose R is a binary relation on a universe U. A pair of approximation operators, $\underline{apr}_R$, $\overline{apr}_R : 2^U \longrightarrow 2^U$, is defined by:*

$$\begin{aligned}
\underline{apr}_R X &= \{x \mid \forall y[xRy \Longrightarrow y \in X]\} \\
&= \{x \mid R_s(x) \subseteq X\}, \\
\overline{apr}_R X &= \{x \mid \exists y[xRy, y \in X]\}. \\
&= \{x \mid R_s(x) \cap X \neq \emptyset\}.
\end{aligned} \tag{4}$$

The system $(2^U, \cap, \cup, \sim, \underline{apr}_R, \overline{apr}_R)$ is called a rough set algebra, where $\cap$, $\cup$, and $\sim$ are set intersection, union, and complement.

For an arbitrary binary relation, if one extends (C2) by substituting $[x]_R$ with $R_s(x)$, the result is not equivalent to that of Definition 1. Our choice of extending (C1) is to ensure that the generalized rough set theory is consistent with modal logic. For the generalized approximation operators, the relationship as expressed by (C3) does not necessarily hold. In Section 3, we we will examine classes of rough set algebras in which (C3) would be valid, and (C1) and (C2) are equivalent.

Consider the upper approximation of a singleton subset $\{y\}$. By definition, we have:

$$\begin{aligned}
\overline{apr}_R\{y\} &= \{x \mid R_s(x) \cap \{y\} \neq \emptyset\} \\
&= \{x \mid y \in R_s(x)\}.
\end{aligned} \tag{5}$$

Therefore,

$$y \in R_s(x) \Longleftrightarrow x \in \overline{apr}_R\{y\}. \tag{6}$$

Applying it to Definition 1, one establishes the relationships between the upper approximations of singleton subsets and approximations of any subset:

$$\begin{array}{ll} \text{(C5)} & \underline{apr}_R X = \{x \mid \forall y[x \in \overline{apr}_R\{y\} \Longrightarrow y \in X]\}, \\ & \overline{apr}_R X = \{x \mid \exists y[x \in \overline{apr}_R\{y\}, y \in X]\}. \end{array}$$

If R is an equivalence relation, (C5) is equivalent to (C3).

Equation (4) provides a construction algorithm that produces a pair of approximation operators from a binary relation. By definition, we have:

$$\begin{array}{ll} \text{(L1)} & \underline{apr}_R U = U, \\ \text{(H1)} & \overline{apr}_R \emptyset = \emptyset; \\ \text{(L2)} & \underline{apr}_R (X \cap Y) = \underline{apr}_R X \cap \underline{apr}_R Y, \\ \text{(H2)} & \overline{apr}_R (X \cup Y) = \overline{apr}_R X \cup \overline{apr}_R Y. \end{array}$$

Pairs of properties with the same number are dual properties. Properties (L1) and (H1) indicate that the two approximation operators are normal [17]. Properties (L2) and (H2) state that the approximation operator $\underline{apr}_R$ is multiplicative, and the approximation operator $\overline{apr}_R$ is additive [42]. One may also say that $\underline{apr}_R$ is distributive with respect to set intersection, and $\overline{apr}_R$ is distributive with respect to set union. They imply the following two properties:

$$\begin{array}{ll} \text{(L3)} & \underline{apr}_R (X \cup Y) \supseteq \underline{apr}_R X \cup \underline{apr}_R Y, \\ \text{(H3)} & \overline{apr}_R (X \cap Y) \subseteq \overline{apr}_R X \cap \overline{apr}_R Y. \end{array}$$

That is, $\underline{apr}_R$ is not distributive with respect to $\cup$, and $\overline{apr}_R$ is not distributive with respect to $\cap$. They may be regarded as being sub-distributive. Unlike standard set-theoretic operators, approximation operators are not truth-functional. The value of an expression containing $\underline{apr}_R$ and $\overline{apr}_R$ may not necessarily be computed from the values of its subexpressions. This implies that approximation operators cannot be expressed through standard set-theoretic operators, such as union, intersection, and complement. They are referred to as non-standard set-theoretic operators. Like set complement, both $\underline{apr}_R$ are $\overline{apr}_R$ are unary operators. A rough set algebra may be viewed as an augment of the Boolean algebra of the power set $(2^U, \cap, \cup, \sim)$ with two additional unary operators $\underline{apr}_R$ and $\overline{apr}_R$. For notational convenience, we assume that operators $\underline{apr}_R$ and $\overline{apr}_R$ have the same priority as $\sim$, and associate from right to left. For example, an expression $\underline{apr}_R \overline{apr}_R \sim X \cap Y$ is equivalent to $(\underline{apr}_R(\overline{apr}_R(\sim(X))) \cap Y)$.

2.2 ALGEBRAIC METHOD

In an algebraic or axiomatic approach, the primitive notion is a system $(2^U, \cap, \cup, \sim, \mathbf{L}, \mathbf{H})$, where $(2^U, \cap, \cup, \sim)$ is the set algebra, and $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$

are unary operators on the power set 2^U. We call $\mathbf{L}$ and $\mathbf{H}$ approximation operators, indicating their intended physical interpretation. They are defined by axioms without direct reference to binary relations.

Definition 2 *Let* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *be two unary operators on the power set* 2^U. *They are dual operators if,*

$$\begin{array}{ll} \text{(L0)} & \mathbf{L}X = \sim\mathbf{H}\sim X, \\ \text{(H0)} & \mathbf{H}X = \sim\mathbf{L}\sim X, \end{array}$$

for all $X \subseteq U$.

Pairs of unary operators which are not dual to each other have been studied by some authors [34,35,49]. In this study, we only consider dual operators. This is consistent with the original proposal of Pawlak [31], in which dual approximation operators are used. By the duality of $\mathbf{L}$ and $\mathbf{H}$, it is sufficient to introduce one operator and to define the other using (L0) or (H0). For example, one may define the operator $\mathbf{L}$ and consider $\mathbf{H}$ as an abbreviation of $\sim\mathbf{L}\sim$. The use of both operators is for the sake of simplicity and clarity.

Iterated use of unary operators can be regarded as composition of operators. For instance, $\mathbf{LH}$ is the composition of $\mathbf{L}$ and $\mathbf{H}$. With three unary operators $\sim$, $\mathbf{L}$, and $\mathbf{H}$, their compositions introduce new operators such as $\sim\mathbf{L}$, $\sim\mathbf{H}$, $\mathbf{L}\sim$, $\mathbf{H}\sim$, $\mathbf{HH}$, $\mathbf{LL}$, $\mathbf{LH}$, and $\mathbf{HL}$. The compositions of $\mathbf{L}$ and $\mathbf{H}$ result in four distinct operators. On the other hand, by the duality of $\mathbf{L}$ and $\mathbf{H}$, it follows:

$$\begin{aligned} &\mathbf{L}\sim X = \sim\mathbf{H}X, \\ &\sim\mathbf{L}X = \mathbf{H}\sim X. \end{aligned} \tag{7}$$

The compositions of $\sim$ with $\mathbf{L}$ and $\mathbf{H}$ indeed introduces only two distinct operators $\mathbf{L}\sim = \sim\mathbf{H}$ and $\sim\mathbf{L} = \mathbf{H}\sim$. In general, one may consider any sequence of unary operators and establish similar properties. From the duality of $\mathbf{L}$ and $\mathbf{H}$, and the double negation law of set complement, one can derive a $\mathbf{L}$-$\mathbf{H}$ interchange rule. In any sequence of adjacent $\mathbf{L}$'s and $\mathbf{H}$'s, $\mathbf{L}$ may be replaced by $\mathbf{H}$ and $\mathbf{H}$ by $\mathbf{L}$ throughout, provided that a $\sim$ is either inserted or deleted both immediately before and immedidately after the sequence. For example, $\mathbf{LMMLM} = \sim\mathbf{MLLML}\sim$.

For the algebraic study of the theory of rough sets, we state the following theorem, which can be easily obtained from the discussion on the constructive approach and a theorem given by Yao [44, pages 298 and 312, Theorem 3].

Theorem 3 *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *are dual unary operators. There exists a binary relation* R *on* U *such that* $\mathbf{L}X = \underline{apr}_R X$ *and* $\mathbf{H}X = \overline{apr}_R X$ *for all* $X \subseteq U$, *if and only if* $\mathbf{L}$ *and* $\mathbf{H}$ *satisfy the properties:*

$$
\begin{array}{ll}
\text{(L1)} & \mathbf{L}U = U, \\
\text{(L2)} & \mathbf{L}(X \cap Y) = \mathbf{L}X \cap \mathbf{L}Y, \\
\text{(H1)} & \mathbf{H}\emptyset = \emptyset, \\
\text{(H2)} & \mathbf{H}(X \cup Y) = \mathbf{H}X \cup \mathbf{H}Y.
\end{array}
$$

Axioms (L1) and (L2) form an independent set of axioms for $\mathbf{L}$, whereas (H1) and (H2) form an independent set for $\mathbf{H}$. By the result of Theorem 3, they are referred to as the basic axioms. A counterpart of Definition 1 is therefore obtained in the algebraic framework.

Definition 4 *Let* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *be a pair of dual operators. If* $\mathbf{L}$ *satisfies axioms* (L1) *and* (L2)*, or equivalently,* $\mathbf{H}$ *satisfies axioms* (H1) *and* (H2)*, the system* $(2^U, \cap, \cup, \sim, \mathbf{L}, \mathbf{H})$ *is called a rough set algebra, and* $\mathbf{L}$ *and* $\mathbf{H}$ *are called approximation operators.*

Set-theoretic operators $\mathbf{L}$ and $\mathbf{H}$ are viewed as the counterparts of the necessity and possibility operators in modal logic [46]. The entire set U corresponds to theses of a modal logic system. Axiom (L1) corresponds to the rule of necessitation of modal logic. Given axiom (L1), (L2) is equivalent to axiom (K):

$$
\text{(K)} \quad \mathbf{L}(\sim X \cup Y) \subseteq \sim \mathbf{L}X \cup \mathbf{L}Y,
$$

which corresponds to the distribution axiom of necessity with respect to the conditional in modal logic [3,19]. Given axiom (H1), (H2) is equivalent to (K′):

$$
\text{(K}'\text{)} \quad \sim \mathbf{H}X \cap \mathbf{H}Y \subseteq \mathbf{H}(\sim X \cap Y).
$$

A rough set algebra characterized by (L1) and (L2) is therefore labeled K by following the convention of labeling modal logic systems [3].

For a finite universe, property (H2) is equivalent to:

$$
\text{(S)} \quad \mathbf{H}X = \bigcup_{x \in X} \mathbf{H}\{x\}.
$$

For the empty set, we define $\mathbf{H}\emptyset = \emptyset$, which is a special case of (S). Approximation operator $\mathbf{H}$ is uniquely determined by its values on singleton subsets of the universe. We can therefore establish the connections between approximations of any subset of U and the upper approximation of the singleton subsets:

$$
\begin{array}{ll}
\text{(A5)} & \mathbf{L}X = \{x \mid \forall y[x \in \mathbf{H}\{y\} \Longrightarrow y \in X]\}, \\
 & \mathbf{H}X = \{x \mid \exists y[x \in \mathbf{H}\{y\}, y \in X]\}.
\end{array}
$$

An operator satisfying axiom (S) is also referred to as an unit operator [42,43]. Operator $\mathbf{L}$ is not additive and $\mathbf{H}$ is not multiplicative. They satisfy a set of weaker conditions:

$$\begin{array}{ll} \text{(L3)} & \mathbf{L}(X \cup Y) \supseteq \mathbf{L}X \cup \mathbf{L}Y, \\ \text{(H3)} & \mathbf{H}(X \cap Y) \subseteq \mathbf{H}X \cap \mathbf{H}Y. \end{array}$$

Such conditions can be equivalently expressed as the monotonicity of $\mathbf{L}$ and $\mathbf{H}$ with respect to set inclusion, namely,

$$\begin{array}{ll} \text{(L4)} & X \subseteq Y \Longrightarrow \mathbf{L}X \subseteq \mathbf{L}Y, \\ \text{(H4)} & X \subseteq Y \Longrightarrow \mathbf{H}X \subseteq \mathbf{H}Y. \end{array}$$

Axiom (L2) implies (L3) and (L4), and axiom (H2) implies (H3) and (H4). The reverse is not true. In fact, they are equivalent to a weaker version of (L2) and (H2):

$$\begin{array}{ll} \text{(L5)} & \mathbf{L}(X \cap Y) \subseteq \mathbf{L}X \cap \mathbf{L}Y, \\ \text{(H5)} & \mathbf{H}(X \cup Y) \supseteq \mathbf{H}X \cup \mathbf{H}Y. \end{array}$$

They are used by Lin and Liu [18] to characterize approximation operators in the framework of neighborhood systems. Such properties have also been studied within the framework of probabilistic rough set models [33,47]. The relationships between various properties are summarized in the following theorem, which can be easily proved.

Theorem 5 *The following implication and equivalence relationships hold between properties of approximation operators:*

$$\begin{array}{ll} \text{(a).} & \text{(L1) and (L2)} \Longleftrightarrow \text{(L1) and (K)}, \\ & \text{(H1) and (H2)} \Longleftrightarrow \text{(H1) and (K}'\text{)}; \\ \text{(b).} & \text{(L2)} \Longrightarrow \text{(L3), (L4), and (L5)}, \\ & \text{(H2)} \Longrightarrow \text{(H3), (H4), and (H5)}; \\ \text{(c).} & \text{(H2)} \Longleftrightarrow \text{(S)}; \\ \text{(d).} & \text{(L3)} \Longleftrightarrow \text{(L4)} \Longleftrightarrow \text{(L5)}, \\ & \text{(H3)} \Longleftrightarrow \text{(H4)} \Longleftrightarrow \text{(H5)}. \end{array}$$

We now turn our attention to composition of unary operators. In a rough set algebra, none of the six unary operators, $\mathbf{LL}$, $\mathbf{HH}$, $\mathbf{HL}$, $\mathbf{LH}$, $\sim\mathbf{L} = \mathbf{H}\sim$, and $\sim\mathbf{H} = \mathbf{L}\sim$, is equivalent to one of $\sim$, $\mathbf{L}$, and $\mathbf{H}$. This suggests that iterated use of the approximation operators is not reducible. In fact, by the duality of $\mathbf{L}$ and $\mathbf{H}$, they define three pairs of dual operators.

The operators $\mathbf{LL}$ and $\mathbf{HH}$ are dual to each other, namely, $\mathbf{LL} = \sim\mathbf{HH}\sim$. By the properties of $\mathbf{L}$ and $\mathbf{H}$, it follows that $\mathbf{LL}$ and $\mathbf{HH}$ satisfy axioms (L1), (L2), (U1), and (U2):

$$\begin{aligned}
&\mathbf{LL}U = U,\\
&\mathbf{HH}\emptyset = \emptyset;\\
&\mathbf{LL}(X \cap Y) = \mathbf{LL}X \cap \mathbf{LL}Y,\\
&\mathbf{HH}(X \cup Y) = \mathbf{HH}X \cup \mathbf{HH}Y.
\end{aligned}$$

Therefore, the system $(2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH})$ is also a rough set algebra. By using the same argument, one may obtain many pairs of approximation operators and the corresponding rough set algebras. One operator is obtained by iterated use of $\mathbf{L}$, and the other by iterated use of $\mathbf{H}$ having the same length. For example, the system $(2^U, \cap, \cup, \sim, \mathbf{LLL}, \mathbf{HHH})$ is also a rough set algebra.

Two operators $\mathbf{HL}$ and $\mathbf{LH}$ are dual to each other. They satisfy axioms (L3) and (H3):

$$\begin{aligned}
&\mathbf{HL}(X \cap Y) \subseteq \mathbf{HL}X \cup \mathbf{HL}Y,\\
&\mathbf{LH}(X \cup Y) \supseteq \mathbf{LH}X \cap \mathbf{LH}Y.
\end{aligned}$$

However, they do not satisty axioms (L1), (H1), (L2), and (H2). The system $(2^U, \cap, \cup, \sim, \mathbf{HL}, \mathbf{LH})$ is weaker than a rough set algebra.

The remaining two operators $\sim\mathbf{L}$ and $\sim\mathbf{H}$ are another pair of dual operators. From de Morgan's laws, and axioms (L1), (L2), (H1), and (H2) of $\mathbf{L}$ and $\mathbf{H}$, we have:

$$\begin{array}{ll}
(\sim\text{L1}) & \sim\mathbf{L}U = \emptyset,\\
(\sim\text{H1}) & \sim\mathbf{H}\emptyset = U;\\
(\sim\text{L2}) & \sim\mathbf{L}(X \cap Y) = \sim\mathbf{L}X \cup \sim\mathbf{L}Y,\\
(\sim\text{H2}) & \sim\mathbf{H}(X \cup Y) = \sim\mathbf{H}X \cap \sim\mathbf{H}Y.
\end{array}$$

It can be verified $\sim\mathbf{L}$ and $\sim\mathbf{H}$ satisfy axioms (∼L1), (∼L2), (∼H1), and (∼H1), if and only if $\mathbf{L}$ and $\mathbf{H}$ satisfy axioms (L1), (L2), (H1), and (H2). The system $(2^U, \cap, \cup, \sim, \sim\mathbf{L}, \sim\mathbf{H})$ can be considered as an alternative representation of rough set algebras.

2.3 CONNECTIONS BETWEEN CONSTRUCTIVE AND ALGEBRAIC APPROACHES

The connections between constructive and axiomatic approaches may be established by identifying axioms of **L** and **H**, under which the system is equivalent to a rough set algebra as defined by a binary relation.

The pair of approximation operators plays a similar role as the pair of necessity and possibility operators in modal logic [44,46]. The theory of rough sets extends set theory in the same way that modal logic extends classical logic. From the results of modal logic [3], various rough set algebras may be defined immediately by using binary relations with additional properties. In particular, the following five properties of binary relation may be used:

$$\begin{aligned}
\text{serial} &\Longleftrightarrow \forall x \exists y [xRy], \\
&\Longleftrightarrow \forall x \exists y [y \in R_s(x)], \\
&\Longleftrightarrow \forall x [R_s(x) \neq \emptyset]; \\
\text{reflexive} &\Longleftrightarrow \forall x [xRx], \\
&\Longleftrightarrow \forall x [x \in R_s(x)]; \\
\text{symmetric} &\Longleftrightarrow \forall x, y [xRy \Longrightarrow yRx], \\
&\Longleftrightarrow \forall x, y [x \in R_s(y) \Longrightarrow y \in R_s(x)]; \\
\text{transitive} &\Longleftrightarrow \forall x, y, z [(xRy, yRz) \Longrightarrow xRz], \\
&\Longleftrightarrow \forall x, y [y \in R_s(x) \Longrightarrow R_s(y) \subseteq R_s(x)]; \\
\text{Euclidean} &\Longleftrightarrow \forall x, y, z [(xRy, xRz) \Longrightarrow yRz], \\
&\Longleftrightarrow \forall x, y [y \in R_s(x) \Longrightarrow R_s(x) \subseteq R_s(y)].
\end{aligned}$$

If the binary is serial, one obtains a rough set algebra that is an interval structure [41]. If the binary relation is reflexive and transitive, one obtains a topological rough set algebra [15,44]. If the binary relation is an equivalence relation, a Pawlak rough set algebra is obtained.

In modal logics, different types of binary relations on the set of possible worlds imply different properties of the necessity and possibility operators. With respect to axioms in modal logic [3], if necessity $\Box$ is replaced by **L**, possibility $\Diamond$ by **H**, negation $\neg$ by set complement $\sim$, conjunction $\wedge$ by set intersection $\cap$, disjunction $\vee$ by set union $\cup$, and implication $\rightarrow$ by set inclusion $\subseteq$, the corresponding axioms are obtained for rough set algebra. By adopting the axioms used by Chellas [3] for the study of modal logic, we have the following list of axioms of approximation operators:

(D) $\quad \mathbf{L}X \subseteq \mathbf{H}X;$

(T)	$\mathbf{L}X \subseteq X$,
(T′)	$X \subseteq \mathbf{H}X$;
(B)	$X \subseteq \mathbf{LH}X$,
(B′)	$\mathbf{HL}X \subseteq X$;
(4)	$\mathbf{L}X \subseteq \mathbf{LL}X$,
(4′)	$\mathbf{HH}X \subseteq \mathbf{H}X$;
(5)	$\mathbf{H}X \subseteq \mathbf{LH}X$,
(5′)	$\mathbf{HL}X \subseteq \mathbf{L}X$.

The dual axioms can be obtained by using (L0), (H0), and the properties of set-theoretic operators. An approximation operator obeys an axiom if and only if the dual operator obeys the dual axiom. For convenience, in subsequent discussion we will use an axiom and its dual interchangeably.

Each axiom corresponds to a property of lower and upper approximation operators constructed from a binary relation having a particular property [46]. Relationships between operators defined by axiomatic and constructive approaches are summarized below.

Theorem 6 *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *is a pair of dual approximation operators, i.e.,* $\mathbf{L}$ *satisties axioms* (L1) *and* (L2), $\mathbf{H}$ *satisties axioms* (H1) *and* (H2), *and they are related by* $\mathbf{H} = \sim\mathbf{L}\sim$. *There exists*

(a). *a serial relation on* U,
(b). *a reflexive relation on* U,
(c). *a symmetric relation on* U,
(d). *a transitive relation on* U,
(e). *an Euclidean relation on* U,

such that $\mathbf{L}X = \underline{apr}_R X$ *and* $\mathbf{H}X = \overline{apr}_R X$ *for all* $X \subseteq U$, *if and only if* $\mathbf{L}$ *and* $\mathbf{H}$ *satisfy axiom:*

(a). (D);
(b). (T′);
(c). (B′);
(d). (4′);
(e). (5′).

This theorem can be proved by following the proof of a theorem given by Yao [44, pages 299 and 313, Theorem 5]. In this theorem, the corresponding pair of condition and conclusion is linked together by the same letter. For instance, one can conclude that there exists a serial binary relation R such that $\mathbf{L}X = \underline{apr}_R X$ and $\mathbf{H}X = \overline{apr}_R X$ for all $X \subseteq U$, if and only if the pair of approximation operators $\mathbf{L}$ and $\mathbf{H}$ satisfies axiom (D). One may state an equivalent theorem using dual axioms. An important implication of the theorem is that one can formulate rough set algebras axiomatically. Since there is an one-to-one correspondence between axioms of operators and axioms of binary relations, we may use either the properties of binary relations or the axioms of operators to label particular classes of rough set algebras.

3 SPECIAL CLASSES OF ROUGH SET ALGEBRAS

In characterizing approximation operators, several axioms are expressed in terms of relationships between approximation operators and the composition of approximation operators. They show the connections between the following three systems:

$$\begin{aligned} \mathrm{S}_L &= (2^U, \cap, \cup, \sim, \mathbf{L}, \mathbf{H}), \\ \mathrm{S}_{HL} &= (2^U, \cap, \cup, \sim, \mathbf{HL}, \mathbf{LH}), \\ \mathrm{S}_{LL} &= (2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH}). \end{aligned} \tag{8}$$

Therefore, in subsequent discussions we will focus on these systems. By choosing different sets of axioms, one may derive different classes of rough set algebras, in which the three systems are either related to each other or reduce to the same system.

3.1 SERIAL ROUGH SET ALGEBRAS (INTERVAL STRUCTURES)

Axiom (D) states that $\mathbf{L}X$ is a subset of $\mathbf{H}X$. This property motivates our definition of lower and upper approximation operators.

Definition 7 *Let $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ be a pair of dual approximation operators. They are called lower and upper approximation operators if axiom* (D) *holds.*

Wybraniec-Skardowska [42] defined upper approximation operators by the condition $\mathbf{H}X \subseteq U$ and axiom (S), and the lower approximation operators by the condition $\mathbf{L}X \subseteq U$ and,

$$\mathbf{L}_w X = \{x \mid \mathbf{H}\{x\} \neq \emptyset, \mathbf{H}\{x\} \subseteq X\}. \tag{9}$$

Conditions $\mathbf{H}X \subseteq U$ and $\mathbf{L}X \subseteq U$ are not explicitly stated in our definition, as they are embodied in the assumption that $\mathbf{L}$ and $\mathbf{H}$ are mappings from 2^U

to 2^U. The lower and upper approximation operators as defined Wybraniec-Skardowska do not necessarily satisfy axiom (D). Unlike our definition, they are not a pair of dual operators.

A rough set algebra obtained by adding only axiom (D) to the basic axioms (L1) and (L2) may be labeled KD. It is equivalent to an algebra constructed from a serial binary relation, and is the weakest one that allows lower and upper approximation operators. In a KD rough set algebra, we have:

$$\begin{array}{ll} \text{(L6)} & \mathbf{L}\emptyset = \emptyset, \\ \text{(H6)} & \mathbf{H}U = U; \\ \text{(L7)} & \mathbf{L}X \cap \mathbf{L}{\sim}X = \emptyset, \\ \text{(H7)} & \mathbf{H}X \cup \mathbf{H}{\sim}X = U. \end{array}$$

The empty set and the universe are exactly represented in the sense that their lower and upper approximations are the same as themselves. For an arbitrary subset of universe, this is not necessarily true. Nevertheless, the lower approximations of a set and its complement are disjoint, and the upper approximations of a set and its complement cover the entire universe. Given axiom (L2), (L6), (L7), and (D) are equivalent. Similarly, (H6), (H7), and (D) are equivalent provided that (H2) holds. The set of axioms consisting of (L1), (L2), and (L6) are in fact used to axiomatize interval structures [41]. In the axiomatization of KD rough set algebras, (D) can be replaced by the condition expressed in terms of the upper approximations of the singleton subsets of U:

$$\text{(d)} \qquad \forall x \exists y [x \in \mathbf{H}\{y\}].$$

Axiom (d) is a converse of the condition of a serial binary relation. By this axiom, the family $\{\mathbf{H}\{x\} \mid x \in U\}$ forms a covering of the universe, i.e., $\bigcup\{\mathbf{H}\{x\} \mid x \in U\} = U$.

In a KD rough set algebra, approximation operators **LL** and **HH** obey axiom (D). The system $(2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH})$ is also a KD rough set algebra. Operator **HL** obeys axioms (L1) and (L6), and operator **LH** obeys axioms (H1) and (H6). They do not satisfy axioms (L2) and (H2). Thus, the system $(2^U, \cap, \cup, \sim, \mathbf{HL}, \mathbf{LH})$ is not a rough set algebra. By axiom (D) and monotonicity of **L** and **H**, we have the following relationships between various approximation operators:

$$\begin{aligned} &\mathbf{L}X \subseteq \mathbf{H}X, \\ &\mathbf{LL}X \subseteq \mathbf{LH}X \subseteq \mathbf{HH}X, \\ &\mathbf{LL}X \subseteq \mathbf{HL}X \subseteq \mathbf{HH}X. \end{aligned} \qquad (10)$$

K rough set algebras do not have such properties.

Yao, Y.Y., Constructive and algebraic methods of the theory of rough sets
Information Sciences , Vol. 109, No. 1-4, pp. 21-47, 1998.

3.2 REFLEXIVE ROUGH SET ALGEBRAS

By adding axiom (T), one obtains KT rough set algebras. Axiom (T′) shows that the upper approximation operator is extensive. The notion of reflexive rough set algebras is related to extension algebras, i.e., a Boolean algebra with added operators satisfying the similar properties [16].

According to (T), a set $X \subseteq U$ lies between its lower and upper approximations, namely, $\mathbf{L}X \subseteq X \subseteq \mathbf{H}X$. Thus, (D) holds in a KT rough set algebra. Given axiom (H2), (T′) can be equivalently expressed as:

$$(\mathrm{t}') \qquad \forall x[x \in \mathbf{H}\{x\}].$$

If $\mathbf{H}\{x\}$ is substituted by $R_s(x)$, one can see its similarity to the condition of a reflexive relation.

In a KT rough set algebra, $\mathbf{LL}$ satisfies (T) and $\mathbf{HH}$ satisfies (T′). The system $(2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH})$ is also a KD rough set algebra. The system $(2^U, \cap, \cup, \sim, \mathbf{HL}, \mathbf{LH})$ is still not a rough set algebra because $\mathbf{HL}$ and $\mathbf{LH}$ do not satisfy axioms (L2) and (H2). For three systems S_L, S_{HL}, and S_{LL}, one can establish the relationships that are stronger than those in equation (10):

$$\begin{aligned}
&\mathbf{LL}X \subseteq \mathbf{L}X \subseteq X \subseteq \mathbf{H}X \subseteq \mathbf{HH}X,\\
&\mathbf{LL}X \subseteq \mathbf{L}X \subseteq \mathbf{LH}X \subseteq \mathbf{H}X \subseteq \mathbf{HH}X,\\
&\mathbf{LL}X \subseteq \mathbf{L}X \subseteq \mathbf{HL}X \subseteq \mathbf{H}X \subseteq \mathbf{HH}X.
\end{aligned} \tag{11}$$

Operators $\mathbf{L}$ and $\mathbf{H}$ produce tighter approximations than that of $\mathbf{LL}$ and $\mathbf{HH}$.

Axioms (T) and (T′) relate a set to its lower and upper approximation. Together with the monotonicity of $\mathbf{L}$ and $\mathbf{H}$, they imply the following properties showing the connections between two sets and their approximations, in addition to (L4) and (H4):

$$\begin{array}{rl}
(\mathrm{L8}) & X \subseteq \mathbf{L}Y \Longrightarrow \mathbf{L}X \subseteq \mathbf{L}Y,\\
(\mathrm{H8}) & \mathbf{H}X \subseteq Y \Longrightarrow \mathbf{H}X \subseteq \mathbf{H}Y;\\
(\mathrm{L9}) & \mathbf{L}X \subseteq Y \Longrightarrow \mathbf{L}X \subseteq \mathbf{H}Y,\\
(\mathrm{H9}) & X \subseteq \mathbf{H}Y \Longrightarrow \mathbf{L}X \subseteq \mathbf{H}Y;\\
(\mathrm{L10}) & \mathbf{L}X \subseteq \mathbf{L}Y \Longrightarrow \mathbf{L}X \subseteq Y,\\
(\mathrm{H10}) & \mathbf{H}X \subseteq \mathbf{H}Y \Longrightarrow X \subseteq \mathbf{H}Y;\\
(\mathrm{L11}) & \mathbf{H}X \subseteq \mathbf{L}Y \Longrightarrow \mathbf{H}X \subseteq Y,\\
(\mathrm{H11}) & \mathbf{H}X \subseteq \mathbf{L}Y \Longrightarrow X \subseteq \mathbf{L}Y.
\end{array}$$

Property (L8) was given by Wybraniec-Skardowska [42]. These properties used eight out of nine possible combinations of two sets and their approximations. For the combination $\mathbf{L}X \subseteq \mathbf{H}Y$, one cannot state similar properties.

Pomykala [35] defined lower approximation operators by a single axiom (T), and upper approximation operators by a single axiom (T′). According to Theorem 6, such an extended system may not be related to a rough set algebra constructed from a binary relation. By adding (L1) and (L2) to (T), one may introduce a more restricted definition of lower and upper approximations. This would require that the binary relation is reflexive, which is stronger than the condition of a serial relation.

3.3 SYMMETRIC ROUGH SET ALGEBRAS

By adding axiom (B), we obtain KB rough set algebras. Axioms (H2) and (B′) imply axiom (H1). Given axiom (H2), (B′) can be equivalently expressed by:

$$\text{(b}'\text{)} \quad \forall x, y[x \in \mathbf{H}\{y\} \Longrightarrow y \in \mathbf{H}\{x\}], \tag{12}$$

which is similar to the condition on $R_s(x)$ of a symmetric binary relation. According to (b′), we have:

$$\begin{aligned} x \in \mathbf{H}X &\Longleftrightarrow \exists y[x \in \mathbf{H}\{y\}, y \in X] \\ &\Longleftrightarrow \exists y[y \in \mathbf{H}\{x\}, y \in X] \\ &\Longleftrightarrow X \cap \mathbf{H}\{x\} \neq \emptyset. \end{aligned} \tag{13}$$

From the duality of $\mathbf{L}$ and $\mathbf{H}$, it follows:

$$x \in \mathbf{L}X \Longleftrightarrow \mathbf{H}\{x\} \subseteq X. \tag{14}$$

Therefore, lower and upper approximation operators can be expressed as:

$$\begin{aligned} \text{(A3)} \qquad & \mathbf{L}X = \{x \mid \mathbf{H}\{x\} \subseteq X\}, \\ & \mathbf{H}X = \{x \mid \mathbf{H}\{x\} \cap X \neq \emptyset\}. \end{aligned}$$

They are an extension of approximation operators (C3) as defined by an equivalence relation.

In a KB rough set algebra, the following relationships hold between two sets and their approximations:

(L12) $X \subseteq \mathbf{L}Y \Longrightarrow \mathbf{H}X \subseteq Y$,
(H12) $\mathbf{H}X \subseteq Y \Longrightarrow X \subseteq \mathbf{L}Y$.

They are equivalent to the condition:

$$X \subseteq \mathbf{L}Y \Longleftrightarrow \mathbf{H}X \subseteq Y,$$

which is in turn equivalent to the condition given by equation (14).

Pomykala [35] showed that if the family $\{\mathbf{H}\{x\} \mid x \in U\}$ is a partition of the universe U, then the first equation in (A3) holds. However, it is not a necessary condition. Wybraniec-Skardowska [42] stated that axioms (S) and (b′) are sufficient. It can be easily verified that if second equations in (A3) holds, $\mathbf{H}$ must satisfy axioms (H2) and (b′), which are equivalent to (S) and (B′). In summary, we have the following theorem.

Theorem 8 *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *is a pair of dual operators. Equations in* (A3) *hold if only if* $\mathbf{H}$ *satisfies axioms* (H2) *and* (B′)*.*

An implication of the theorem is that a KB rough set algebra is the weakest one that allows expressions (A3). Wybraniec-Skardowska [42] claimed that axioms (S) and (b′) on $\mathbf{H}$ are sufficient for $\mathbf{H}$ and $\mathbf{L}_w$, defined by equation (9), to be a pair of dual operators. This is in fact incorrect. In a KB rough set algebra, operator $\mathbf{L}$ is not necessarily equal to $\mathbf{L}_w$. In the light of Theorem 8, an additional condition, $\mathbf{H}\{x\} \neq \emptyset$, for all $x \in U$, is required in order for $\mathbf{H}$ and $\mathbf{L}_w$ to be dual operators. This stems from the extra condition $\mathbf{H}\{x\} \neq \emptyset$ in the definition of operator $\mathbf{L}_w$.

In a KB rough set algebra, from (B) we have:

$$\mathbf{H}X \subseteq \mathbf{LH}(\mathbf{H}X). \tag{15}$$

By combining with the monotonicity of $\mathbf{L}$, it follows:

$$\mathbf{LH}X \subseteq \mathbf{LL}\,\mathbf{HH}X. \tag{16}$$

By (B) again, we have:

$$X \subseteq \mathbf{LL}\,\mathbf{HH}X. \tag{17}$$

Thus, $(2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH})$ is also a KB rough set algebra. The system $(2^U, \cap, \cup, \sim, \mathbf{HL}, \mathbf{LH})$ is still not a rough set algebra. Axioms (B) and (B′) state the relationship between X and its images by operators $\mathbf{HL}$ and $\mathbf{LH}$, i.e., $\mathbf{HL}X \subseteq X \subseteq \mathbf{LH}X$. No relationships between three systems S_L, S_{HL}, and S_{LL} can be exstablished.

3.4 TRANSITIVE ROUGH SET ALGEBRAS

A transitive rough set algebra is characterized by adding axiom (4), which is denoted by K4. Given axiom (H2), axiom (4′) may be equivalently expressed as [42]:

$$\text{(iv}'\text{)} \qquad \forall x, y[y \in \mathbf{H}\{x\} \Longrightarrow \mathbf{H}\{y\} \subseteq \mathbf{H}\{x\}],$$

which is similar to the condition of a transitive relation. Two additional equivalent expressions are:

$$\begin{array}{ll} \text{(iv}''\text{)} & \forall x[\mathbf{HH}\{x\} \subseteq \mathbf{H}\{x\}], \\ \text{(iv}'''\text{)} & \forall x, y, z[(y \in \mathbf{H}\{x\}, z \in \mathbf{H}\{y\}) \Longrightarrow z \in \mathbf{H}\{x\}]. \end{array}$$

By monotonicity and axioms (4) and (4′), we have properties:

$$\begin{array}{ll} \text{(L13)} & \mathbf{L}X \subseteq Y \Longrightarrow \mathbf{L}X \subseteq \mathbf{L}Y, \\ \text{(H13)} & X \subseteq \mathbf{H}Y \Longrightarrow \mathbf{H}X \subseteq \mathbf{H}Y. \end{array} \tag{18}$$

In fact, (H13) may be viewed as another representation of (iv′) using arbitrary subsets of U, instead of singleton subsets.

Axioms (4) and (4′) state the connections between $\mathbf{L}$ and $\mathbf{LL}$, and between $\mathbf{H}$ and $\mathbf{HH}$, namely, $\mathbf{L}X \subseteq \mathbf{LL}X$ and $\mathbf{HH}X \subseteq \mathbf{H}$. It can be easily verified that $\mathbf{LL}$ and $\mathbf{HH}$ satisfy (4) and (4′). The system $(2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH})$ is also a K4 rough set algebra. It produces tighter approximations. The system $(2^U, \cap, \cup, \sim, \mathbf{HL}, \mathbf{LH})$ is still not a rough set algebra.

3.5 REFLEXIVE AND SYMMETRIC ROUGH SET ALGEBRAS

A reflexive and symmetric rough set algebra is characterized by axioms (L2), (T), and (B), which is denoted by KTB. Axiom (L1) follows from (L2) and (B). It corresponds to a rough set algebra constructed from a tolerance relation, i.e., a reflexive and symmetric relation. Its modal logic counterpart is also commonly known as the system B.

KTB rough set algebras are perhaps the best studied extension of rough sets [23,34,49]. By combining axioms (T) and (B), we have $\mathbf{L}X \subseteq \mathbf{LH}X$, which can in fact be obtained from axiom (T′) and the monotonicity of $\mathbf{L}$. In other words, the combination of (T) and (B) does not introduce additional properties in the sense any property of KTB must be a property of either KT or KB. By combining equation (11) and axioms (B) and (B′), we have the following relationship:

$$\mathbf{LL}X \subseteq \mathbf{L}X \subseteq \mathbf{HL}X \subseteq X \subseteq \mathbf{LH}X \subseteq \mathbf{H}X \subseteq \mathbf{HH}X. \tag{19}$$

That is, for each subset of U three systems S_L, S_{HL}, and S_{LL} produce a nested family of approximations.

3.6 REFLEXIVE AND TRANSITIVE (TOPOLOGICAL) ROUGH SET ALGEBRAS

A KT4 rough set algebra is defined by axioms (L1), (L2), (T), and (4). It corresponds to the modal logic system S4. Algebraically speaking, KT4 rough set algebras are related to the notion of topological Boolean algebras [37], or simply closure algebras by some authors [21].

Given axioms (T) and (T′), (4) and (4′) can be replaced by:

$$\begin{array}{ll} \text{(L14)} & \mathbf{L}X = \mathbf{LL}X, \\ \text{(H14)} & \mathbf{H}X = \mathbf{HH}X. \end{array}$$

They are special properties of KT4 in the sense that they do not hold in a KT and a K4 rough set algebra. They were used in the study of approximation operators by some authors [15,18]. These properties can be considered as reduction laws, which enable us to replace a sequence of approximation operators by a shorter sequence. In other words, two systems $(2^U, \cap, \cup, \sim, \mathbf{L}, \mathbf{H})$ and $(2^U, \cap, \cup, \sim, \mathbf{LL}, \mathbf{HH})$ become the same system. The relationship to the other system is given by:

$$\begin{array}{l} \mathbf{LL}X = \mathbf{L}X \subseteq X \subseteq \mathbf{H}X = \mathbf{HH}X, \\ \mathbf{LL}X = \mathbf{L}X \subseteq \mathbf{LH}X \subseteq \mathbf{H}X = \mathbf{HH}X, \\ \mathbf{LL}X = \mathbf{L}X \subseteq \mathbf{HL}X \subseteq \mathbf{H}X = \mathbf{HH}X. \end{array} \tag{20}$$

Axioms (L1), (L2), (T), and (L14) of $\mathbf{L}$, and (H1), (H2), (T′), and (H14) of $\mathbf{H}$ are the Kuratowski axioms of interior and closure operators of a topological space. Such an algebra is therefore referred to as a topological rough set algebra. A subset X of U is said to be open if $\mathbf{L}X = X$, and closed if $\mathbf{H}X = X$. It follows from axioms (L14) and (H14) that the lower approximation of any subset of X is an open set, and the upper approximation is a closed set. Moreover, $\mathbf{L}X$ is the greatest open subset of X, and $\mathbf{H}X$ is the least closed set containing X. This conform to the original definition of Pawlak [31,32].

Closure operators weaker than topological closure operators have been investigated by many authors [4,9,42]. For example, a closure operator can be defined by a set of axioms consisting of (H4), (T′), and (H14):

$$\begin{array}{rl} \text{(H4)} & X \subseteq Y \Longrightarrow \mathbf{H}X \subseteq \mathbf{H}Y, \\ \text{(T}'\text{)} & X \subseteq \mathbf{H}X, \\ \text{(H14)} & \mathbf{H}X = \mathbf{HH}X. \end{array}$$

They can be equivalently expressed by a single axiom and its dual axiom [42]:

$$\begin{array}{rl} \text{(L15)} & \mathbf{L}X \subseteq Y \Longleftrightarrow \mathbf{LX} \subseteq \mathbf{L}Y, \\ \text{(H15)} & X \subseteq \mathbf{H}Y \Longleftrightarrow \mathbf{HX} \subseteq \mathbf{H}Y. \end{array}$$

They are the combinations of (L10) and (L13), and (H10) and (H13), which are properties of KT and K4 rough set algebras, respectively.

3.7 SYMMETRIC AND TRANSITIVE ROUGH SET ALGEBRAS

The symmetric and transitive rough set algebras, denoted KB4, are characterized by axioms (L2), (B), and (4). In such an algebra, (5) and (5′) hold, which are neither axioms of KB nor axioms of K4. By monotonicity, (5), and (5′), we have:

$$\begin{array}{rl} \text{(L16)} & X \subseteq \mathbf{L}Y \Longrightarrow \mathbf{H}X \subseteq \mathbf{L}Y, \\ \text{(H16)} & \mathbf{H}X \subseteq Y \Longrightarrow \mathbf{H}X \subseteq \mathbf{L}Y. \end{array} \qquad (21)$$

Given axiom (H2), (5′) can be equivalently expressed as:

$$\text{(v}'\text{)} \qquad \forall x, y, z[(x \in \mathbf{H}\{y\}, x \in \mathbf{H}\{z\}) \Longrightarrow y \in \mathbf{H}\{z\}].$$

Given (b′), (v′) can be reexpressed by:

$$\text{(v}''\text{)} \qquad \forall x, y[y \in \mathbf{H}\{x\} \Longrightarrow \mathbf{H}\{x\} \subseteq \mathbf{H}\{y\}],$$

which is related to the condition of an Euclidean relation.

Consider two elements $x, y \in U$. If $x \in \mathbf{H}\{y\}$, it follows from (b′) that $y \in \mathbf{H}\{x\}$. By (iv′), one can conclude that $\mathbf{H}\{x\} = \mathbf{H}\{y\}$, $x \in \mathbf{H}\{x\}$, and $y \in$

$\mathbf{H}\{y\}$. The family $\{\mathbf{H}\{x\} \mid x \in U\}$ consists of pairwise disjoint subsets of U. It is not necessarily a partition of the universe. By adding the set $\{x \mid \mathbf{H}\{x\} = \emptyset\}$, we have a partition of the universe:

$$\{\mathbf{H}\{x\} \mid x \in U\} \cup \{x \mid \mathbf{H}\{x\} = \emptyset\}. \tag{22}$$

From the above observations, it follows:

$$\begin{aligned} x \in \mathbf{H}X &\Longleftrightarrow \exists y[x \in \mathbf{H}\{y\}, y \in X] \\ &\Longleftrightarrow \exists y[x \in \mathbf{H}\{y\}, y \in \mathbf{H}\{y\}, y \in X] \\ &\Longleftrightarrow \exists y[x \in \mathbf{H}\{y\}, y \in \mathbf{H}\{y\} \cap X \neq \emptyset] \\ &\Longleftrightarrow \exists y[x \in \mathbf{H}\{y\}, \mathbf{H}\{y\} \cap X \neq \emptyset]. \end{aligned} \tag{23}$$

By the duality of $\mathbf{L}$ and $\mathbf{H}$ and the fact that $\mathbf{H}\{x\}$'s are pairwise disjoint, one obtains:

$$\begin{aligned} \text{(A6)} \quad \mathbf{L}X &= \Big(\bigcup\{\mathbf{H}\{x\} \mid \mathbf{H}\{x\} \subseteq X\}\Big) \cup \{x \mid \mathbf{H}\{x\} = \emptyset\}, \\ \mathbf{H}X &= \bigcup\{\mathbf{H}\{x\} \mid \mathbf{H}\{x\} \cap X \neq \emptyset\}. \end{aligned} \tag{24}$$

Although the second expression of (A6) is the same as that of (C4), the first expression has an extra component. The following theorem gives the necessary and sufficient condition under which (A6) holds.

Theorem 9 *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *is a pair of dual operators. Equations in* (A6) *hold if only if* $\mathbf{H}$ *satisfies axioms* (H2)*,* (B′)*, and* (4′)*.*

This theorem suggests that KB4 rough set algebras are the weakest algebras that allow expression (A6).

3.8 PAWLAK ROUGH SET ALGEBRAS

Axioms (T), (B), (4), and (5) do not form a set of independent axioms, although they are pairwise independent. One independent set consists of (T), (B), and (4), and another independent set consists of (T) and (5). Adding these axioms to (L1) and (L2) results in the class of Pawlak rough set algebras, which is a counterpart of modal logic system S5. This class can be defined axiomatically by either a set of axioms consisting of (L2), (T), (B), and (4), denoted by KTB4, or another set consisting of (L2), (T), and (5), denoted by KT5. Axioms (5) and (T) imply (B), and axioms (L2) and (B) implies axiom (L1). Axiom (L1) is indeed not needed although it is explicitly stated in some studies [18,44]. Axiom (t′) implies that the family $\{\mathbf{H}\{x\} \mid x \in U\}$

forms a covering of U. Axioms (b) and (4′) imply that $\mathbf{H}\{x\}$'s are pairwise disjoint. Therefore, the family $\{\mathbf{H}\{x\} \mid x \in U\}$ is a partition of the universe.

A Pawlak rough set algebra is a KB rough set algebra and (A3) holds. It is a KB4 rough set algebra and (A6) holds. In fact, a stronger version of (A6) can be used:

$$\begin{aligned} \text{(A4)} \quad & \mathbf{L}X = \bigcup\{\mathbf{H}\{x\} \mid \mathbf{H}\{x\} \subseteq X\}, \\ & \mathbf{H}X = \bigcup\{\mathbf{H}\{x\} \mid \mathbf{H}\{x\} \cap X \neq \emptyset\}. \end{aligned} \tag{25}$$

A set of neccessary and sufficient conditions for (A4) is given in the following theorem.

Theorem 10 *Suppose* $\mathbf{L}, \mathbf{H} : 2^U \longrightarrow 2^U$ *is a pair of dual operators. Equations in* (A4) *hold if only if* $\mathbf{H}$ *satisfies axioms* (H2)*,* (T′)*,* (B′)*, and* (4′)*.*

A Pawlak rough set algebra is a topological rough set algebra. Axioms (L14) and (H14) enable us to reduce **LL** and **HH** to **L** and **H**. Given axioms (T) and (T′), (5) and (5′) can be replaced by:

$$\begin{aligned} \text{(L17)} \quad & \mathbf{H}X = \mathbf{LH}X, \\ \text{(H17)} \quad & \mathbf{L}X = \mathbf{HL}X. \end{aligned}$$

Therefore, the lower and upper approximations of a subset of universe are both open and closed sets. Three systems become the same system. Operators **LL** and **HL** are the same as **L**, and **LH** and **HH** are the same as **H**. We therefore have a stronger rule for reducing approximation operators: in any sequence of approximation operators we may delete all but the last one.

In the axiomatization of Pawlak rough set algebras, Lin and Liu [18] used the duality of **L** and **H**, and a set of axioms consisting of (H1), (H2), (T), (L14), and (H17). Given axioms (H2) and (T), (H17) indeed implies (H1) and (L14). In addition, axioms (T) and (5) are equivalent to (T) and (H17). Therefore, a reduced set of axioms consisting of (H2), (T), and (H17) may be used.

Comer [5,6] adopted the following axioms from the study of cylindric algebras [9]:

$$\begin{aligned} \text{(L18)} \quad & \mathbf{L}(X \cup \mathbf{L}Y) = \mathbf{L}X \cup \mathbf{L}Y, \\ \text{(H18)} \quad & \mathbf{H}(X \cap \mathbf{H}Y) = \mathbf{H}X \cap \mathbf{H}Y. \end{aligned}$$

A Pawlak rough set algebra satisfies these axioms. It is therefore a cylindric algebra of dimension 1. A Pawlak algebra can be equivalently characterized by a set of axioms consisting of (H0), (T), and (H18). They imply (H2), (H14), and (H17).

3.9 SUMMARY AND DISCUSSION

So far, we have examined several important classes of rough set algebras. Theorems 3 and 6 establish a linkage between constructive and axiomatic approaches. Properties of binary relations determine the properties of the constructed approximation operators. For a pair of dual approximation operators satisfying certain axioms, there exists a binary relation which obeys certain properties and produces the same approximation operators. Figure 1 summarizes the main results. The labels inside the rectangular boxes indicate the axioms that characterize the class of rough set algebras, the properties of binary relations are given beside each box. From the discussion, we have some useful observations. Rough set algebras (i.e., K) are equivalent to algebras constructed from binary relations without any additional properties, and hence are the weakest class. In such algebras, (A5) holds. Pawlak rough set algebras (i.e., KT5 or S5) are the strongest class, in which (A4) holds. Symmetric rough set algebras (i.e., KB) are the weakest algebras that allows the relationship as expressed by (A3). Symmetric and transitive rough set algebras (i.e., KB4) are the weakest algebras in which (A6) holds. The combination of KB and KT does not introduce any additional properties which are not in either KB or KT. The combinations of other algebras introduce additional properties. For instance, the combination of KT and K4 produce $\mathbf{L}X = \mathbf{LL}X$, which does not hold in both KT and K4.

Our discussions are based on a fixed construction algorithm:

$$\text{(C1)} \qquad \begin{aligned} \underline{apr}_R X &= \{x \mid \forall y[xRy \Longrightarrow y \in X]\}, \\ \overline{apr}_R X &= \{x \mid \exists y[xRy, y \in X]\}. \end{aligned}$$

It enables us to apply the results from modal logic intermediately. An important implication of a fixed construction method is that the binary relation is uniquely defined. However, the axiomatic approach should not be viewed in this narrow sense. There may exist more than one way to define a binary relation and to design a construction algorithm that will produce the same approximation operators. For a specific binary relation, there are many ways to define approximation operators.

In the characterization of upper approximation operators, axioms (T) to (5) can be expressed in terms of the upper approximations of singleton subsets. Furthermore, axioms (t′) to (iv′) are exactly the same as the conditions of reflexive, symmetric, and transitive relations, expressed in terms of successor

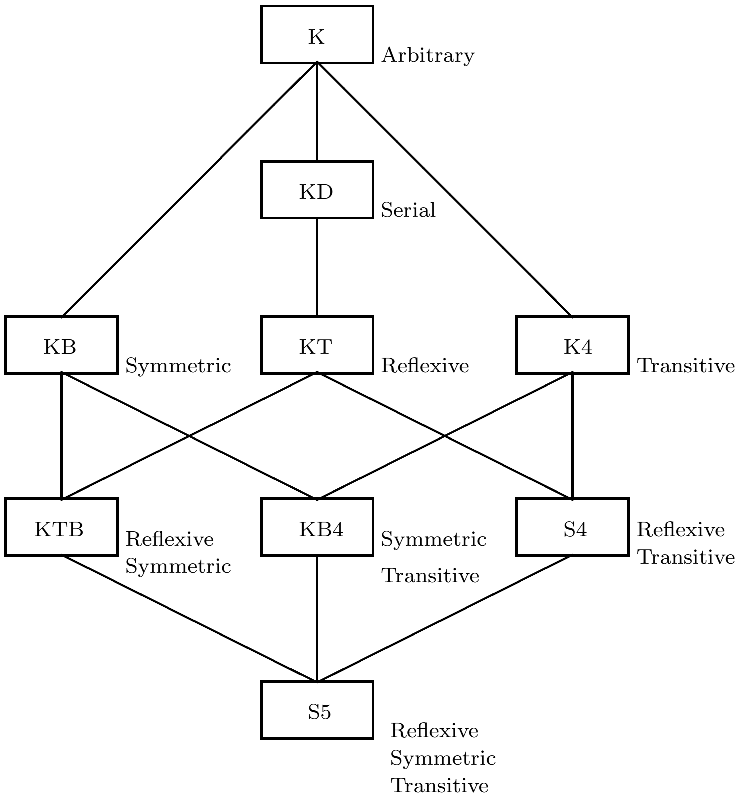

Fig. 1. Classes of Rough Set Algebras

neighborhoods. For example, axiom (t′) and the condition of reflexive relation are given by:

$$\begin{aligned} \text{axiom (t}'\text{)} : &\quad \forall x[x \in \mathbf{H}\{x\}], \\ \text{reflexive relation} : &\quad \forall x[x \in R_s(x)]. \end{aligned}$$

From any one, the other can be derived by exchanging $\mathbf{H}\{x\}$ and $R_s(x)$. Based on such a correspondence, an approximation operator may be defined by:

$$\overline{apr}'_R\{x\} = R_s(x). \tag{26}$$

One can extend $\overline{apr}'$ all subsets of U:

$$\overline{apr}'X = \bigcup_{x \in X} \overline{apr}'\{x\}. \tag{27}$$

They would satisfy the axioms corresponding to reflexive, symmetric, and transitive relations. We have:

$$\begin{aligned}
&\overline{apr}' \text{ satisfies axiom (t}'\text{)} \Longleftrightarrow R \text{ is reflexive,}\\
&\overline{apr}' \text{ satisfies axiom (b}'\text{)} \Longleftrightarrow R \text{ is symmetric,}\\
&\overline{apr}' \text{ satisfies axiom (iv}'\text{)} \Longleftrightarrow R \text{ is transitive.}
\end{aligned} \tag{28}$$

For an arbitrary binary relation R, the operators $\overline{apr}'$ and its dual $\underline{apr}'$ can be expressed in a form similar to (C1) as:

$$\begin{aligned}
\text{(C1}'\text{)} \qquad &\underline{apr}'_R X = \{x \mid \forall y[yRx \Longrightarrow y \in X]\},\\
&\overline{apr}'_R X = \{x \mid \exists y[yRx, y \in X]\}.
\end{aligned}$$

This construction algorithm was used by Orlowska [26]. Construction algorithms (C1) and (C1′) define two distinct rough set algebras, $(2^U, \cap, \cup, \sim, \underline{apr}_R, \overline{apr}_R)$ and $(2^U, \cap, \cup, \sim, \underline{apr}'_R, \overline{apr}'_R)$. For a serial or an Euclidean relation, the two rough set algebras may not be of the same type. For a reflexive or a transitive relation, the two rough set algebras are of the same type. For a symmetric relation, they reduce to the same rough set algebra.

The above discussion can be illustrated by two examples.

Example 11 *Consider a universe $U = \{a, b, c\}$. Let R be a binary relation on U:*

$$aRa, \qquad aRb, \qquad bRb, \qquad cRa, \qquad cRc.$$

The relation is only reflexive. According to construction algorithms (C1) *and* (C1′)*, we can define the following two pair of approximation operators:*

X	$\underline{apr}_R X$	$\overline{apr}_R X$	$\underline{apr}'_R X$	$\overline{apr}'_R X$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a, c\}$	$\emptyset$	$\{a, b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	$\emptyset$	$\{b\}$
$\{c\}$	$\emptyset$	$\{c\}$	$\{c\}$	$\{a, c\}$
$\{a, b\}$	$\{a, b\}$	U	$\{b\}$	$\{a, b\}$
$\{a, c\}$	$\{c\}$	$\{a, c\}$	$\{a, c\}$	U
$\{b, c\}$	$\{b\}$	U	$\{c\}$	U
U	U	U	U	U

Although the derived two pairs of operators are different, they both form KT rough set algebras.

Example 12 *Consider a universe $U = \{a, b, c\}$. Suppose a pair of dual approximation operators is given by:*

$$
\begin{aligned}
&\mathbf{L}\emptyset = \emptyset, && \mathbf{H}\emptyset = \emptyset,\\
&\mathbf{L}\{a\} = \{c\}, && \mathbf{H}\{a\} = \{a, c\},\\
&\mathbf{L}\{b\} = \{b\}, && \mathbf{H}\{b\} = \{a, b\},\\
&\mathbf{L}\{c\} = \emptyset, && \mathbf{H}\{c\} = \emptyset,\\
&\mathbf{L}\{a, b\} = U, && \mathbf{H}\{a, b\} = U,\\
&\mathbf{L}\{a, c\} = \{c\}, && \mathbf{H}\{a, c\} = \{a, c\},\\
&\mathbf{L}\{b, c\} = \{b\}, && \mathbf{H}\{b, c\} = \{a, b\},\\
&\mathbf{L}U = U, && \mathbf{H}U = U.
\end{aligned}
$$

They satisfy axioms (L1), (L2), (H1), (H2), *and* (D), *and do not satisfy other axioms. According to Theorem 6, there exists a serial binary relation producing the same operators. The binary relation can be constructed by:*

$$y \in R_s(x) \Longleftrightarrow x \in \mathbf{H}\{y\}. \tag{29}$$

The binary relation is given by:

$$aRa, \quad aRb, \quad bRb, \quad cRa.$$

The pair of approximation operators computed by using (C1) *is the same as the operators* $\mathbf{L}$ *and* $\mathbf{H}$.

With respect to $\mathbf{L}$ *and* $\mathbf{H}$, *one may define a binary relation by:*

$$R'_s(x) = \mathbf{H}\{x\}. \tag{30}$$

It produce the binary relation:

$$aR'a, \quad aR'c, \quad bR'a, \quad bR'b.$$

Obviously, R' is the converse relation of R. It satisfies a condition that is the converse of the condition of a serial binary relation, namely, for every $x \in U$ there exists a y such that $yR'x$. A pair of operators constructed by using (C1$'$) *is the same as the operators* $\mathbf{L}$ *and* $\mathbf{H}$.

In the constructive approach, the physical meaning of approximation operators is associated with the physical meaning of binary relations by using a construction algorithm. By choosing different constructive algorithms, one may derive other formulations of approximation operators. In contrast, the axiomatic approach focuses on the properties of approximation operators without being restricted to any particular construction algorithm. A pair of approximation operators may be interpreted by using different binary relations and construction algorithms. One may choose a specific relation and algorithm depending on the particular situation. The axiomatic approach therefore provides a more general framework for the study of rough sets. In some situations the binary relations may not be readily available. The physical meaning of binary relations in other mathematical structures, such as Boolean algebra and lattice, may not be entirely clear. The requirement of an explicit binary relation may also make it difficult to extend such a formulation. The axiomatic approach can be easily applied in these situations.

4 CONCLUSION

The theory of rough sets can be developed in at least two ways: constructively and algebraically (axiomatically). They are complementary to each other. The constructive approach is more suitable for practical applications of rough sets, while the algebraic approach is appropriate for studying the structures of rough set algebras. Many studies have focused on the constructive approach, due to its simplicity and associated intuitiveness. An in-depth examination of both approaches will enhance our understanding of the theory of rough sets.

In the constructive approach, a pair of lower and upper approximation operators is defined based on a binary relation. The relation represents the available information of the system. Traditionally, one defines an equivalence relation on a universe of objects using their attribute values. An advantage of such an approach is that every notion has a clear and well understood physical meaning. On the other hand, the algebraic approach deals with axioms that must be satisfied by approximation operators, without explicitly referring to a binary relation. An advantage of the algebraic approach is that one can focus on algebraic properties of approximations. From the same binary relation, one may be able to define different rough set algebras whose approximation operators obey the same set of axioms. The algebraic approach is more flexible and general than the constructive approach.

References

[1] G. Cattaneo, Generalized rough sets: preclusivity fuzzy-intuitionistic (BZ) lattices, *Studia Logica*, **58**, 47-77, 1997.

[2] G. Cattaneo, Mathematical foundations of roughness and fuzziness, manuscript, Dipartimento di Scienze dell'Informazione, Univerita di Milano, via Comelico 39, I-20135 Milano, Italy, 1997.

[3] B.F. Chellas, *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

[4] P.M. Cohn, *Universal Algebra*, Harper and Row Publishers, New York, 1965.

[5] S. Comer, An algebraic approach to the approximation of information, *Fundamenta Informaticae*, **14**, 492-502, 1991.

[6] S. Comer, On connections between information systems, rough sets, and algebraic logic, in: *Algebraic Methods in Logic and Computer Science*, Banach Center Publications 28, 117-124, 1993.

[7] I. Düntsch, Rough sets and algebras of relations, in: *Incomplete Information: Rough Set Analysis*, edited by E. Orlowska, Physica-Verlag, Heidelberg, pp. 95-108, 1998.

[8] M. Gehrke, and E. Walker, On the structure of rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **40**, 235-245, 1992.

[9] L. Henkin, J.D. Monk, and A. Tarski, *Cylindric Algebras I*, North-Holland, Amsterdam, 1971.

[10] G.E. Hughes, and M.J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 1968.

[11] T.B. Iwinski, Algebraic approach to rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **35**, 673-683, 1987.

[12] T.B. Iwinski, Rough orders and rough concepts, *Bulletin of the Polish Academy of Sciences, Mathematics*, **36**, 187-192, 1988.

[13] G.J. Klir, Multivalued logics versus modal logics: alternative frameworks for uncertainty modelling, in: *Advances in Fuzzy Theory and Technology*, volume II, edited by P.P. Wang, Duke University, pp. 3-47, 1994.

[14] G.J. Klir, and B. Yuan, *Fuzzy Sets and Fuzzy Logic, Theory and Applications*, Prentice Hall, New Jersey, 1995.

[15] J. Kortelainen, On relationship between modified sets, topological spaces and rough sets, *Fuzzy Sets and Systems*, **61**, 91-95, 1994.

[16] E.J. Lemmon, An extension algebra and the modal system T, *Notre Dame Journal of Formal Logic*, **1**, 3-12, 1960.

[17] E.J. Lemmon, Algebraic semantics for modal logics I, II, *Journal of Symbolic Logic*, **31**, 46-65, 191-218, 1966.

[18] T.Y. Lin, and Q. Liu, Rough approximate operators: axiomatic rough set theory, in: *Rough Sets, Fuzzy Sets and Knowledge Discovery*, edited by W.P. Ziarko, Springer-Verlag, London, pp. 256-260, 1994.

[19] B. Marchal, Modal logic – a brief tutorial, in: *Non-Standard Logics for Automated Reasoning*, edited by P. Smets, A. Mamdani, D. Dubois, and H. Prade, Academic Press, New York, pp. 15-23, 1988.

[20] J.C.C. McKinsey, A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology, *The Journal of Symbolic Logic*, **6**, 117-134, 1941.

[21] J.C.C. McKinsey, and A. Tarski, The algebra of topology, *Annals of Mathematics*, **45**, 141-191, 1944.

[22] A. Nakamura, and J.M. Gao, A logic for fuzzy data analysis, *Fuzzy Sets and Systems*, **39**, 127-132, 1991.

[23] A. Nakamura, and J.M. Gao, On a KTB-modal fuzzy logic, *Fuzzy Sets and Systems*, **45**, 327-334, 1992.

[24] H.T. Nguyen, Intervals in Boolean rings: approximation and logic, *Foundations of Computer and Decision Sciences*, **17**, 131-138, 1992.

[25] E. Orlowska, Logic of indiscernibility relations, *Bulletin of Polish Academy of Science, Mathematics*, **33**, 475-485, 1985.

[26] E. Orlowska, Kripke semantics for knowledge representation logics, *Studia Logica*, **49**, 255-272, 1990.

[27] E. Orlowska, Rough set semantics for non-classical logics, in: *Rough Sets, Fuzzy Sets and Knowledge Discovery* edited by W.P. Ziarko, Springer-Verlag, London, pp. 143-148, 1994.

[28] P. Pagliani, A pure logic-algebraic analysis of rough top and rough bottom equalities, in: *Rough Sets, Fuzzy Sets and Knowledge Discovery*, edited by W.P. Ziarko, Springer-Verlag, London, pp. 225-241, 1994.

[29] P. Pagliani, Rough sets theory and logic-algebraic structures, in: *Incomplete Information: Rough Set Analysis*, edited by E. Orlowska, Physica-Verlag, Heidelberg, pp. 109-190, 1998.

[30] Z. Pawlak, Information systems, theoretical foundations, *Information Systems*, **6**, 205-218, 1981.

[31] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences*, **11**, 341-356, 1982.

[32] Z. Pawlak, Rough classification, *International Journal of Man-Machine Studies*, **20**, 469-483, 1984.

[33] Z. Pawlak, S.K.M. Wong, and W. Ziarko, Rough sets: probabilistic versus deterministic approach, *International Journal of Man-machine Studies*, **29**, 81-95, 1988.

[34] J.A. Pomykala, Approximation operations in approximation space, *Bulletin of the Polish Academy of Sciences, Mathematics*, **35**, 653-662, 1987.

[35] J.A. Pomykala, On definability in the nondeterministic information system, *Bulletin of the Polish Academy of Sciences, Mathematics*, **36**, 193-210, 1988.

[36] J. Pomykala, and J.A. Pomykala, The Stone algebra of rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **36**, 495-508, 1988.

[37] H. Rasiowa, *An Algebraic Approach to Non-classical Logics*, North-Holland, Amsterdam, 1974.

[38] H. Rasiowa, and R. Sikorski, *Mathematics of Metamathematics*, PWN, Warsaw, 1970.

[39] A. Wasilewska, Topological rough algebras, in: *Rough Sets and Data Mining: Analysis for Imprecise Data*, edited by T.Y. Lin and N. Cercone, Kluwer Academic Publishers, Boston, pp. 411-425, 1997.

[40] A. Wiweger, On topological rough sets, *Bulletin of the Polish Academy of Sciences, Mathematics*, **37**, 89-93, 1989.

[41] S.K.M. Wong, L.S. Wang, and Y.Y. Yao, On modeling uncertainty with interval structures, *Computational Intelligence*, **11**, 406-426, 1995.

[42] U. Wybraniec-Skardowska, On a generalization of approximation space, *Bulletin of the Polish Academy of Sciences, Mathematics*, **37**, 51-61, 1989.

[43] U. Wybraniec-Skardowska, Unit operations, *ZeszytyNaukowe Wyzszej Szkoly Pedagogicznej im Powstancow Slaskich w Opolu, Matematyka*, **XXVII**, 113-129, 1992.

[44] Y.Y. Yao, Two views of the theory of rough sets in finite universes, *International Journal of Approximate Reasoning*, **15**, 291-317, 1996.

[45] Y.Y. Yao, A comparative study of fuzzy sets and rough sets, manuscript, Department of Computer Science, Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1, 1997.

[46] Y.Y. Yao, and T.Y. Lin, Generalization of rough sets using modal logic, *Intelligent Automation and Soft Computing, an International Journal*, **2**, 103-120, 1996.

[47] Y.Y. Yao, and S.K.M. Wong, A decision theoretic framework for approximating concepts, *International Journal of Man-machine Studies*, 37, 793-809, 1992.

[48] W. Zakowski, On a concept of rough sets, *Demonstratio Mathematica*, **XV**, 1129-1133, 1982.

[49] W. Zakowski, Approximations in the space (U, Π), *Demonstratio Mathematica*, **XVI**, 761-769, 1983.