Granular Computing based on Rough Sets, Quotient Space Theory, and Belief Functions

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Abstract. A model of granular computing (GrC) is proposed by reformulating, re-interpreting, and combining results from rough sets, quotient space theory, and belief functions. Two operations, called zooming-in and zooming-out operations, are used to study connections between the elements of a universe and the elements of a granulated universe, as well as connections between computations in the two universes. The operations are studied with respect to multi-level granulation structures.

1 Introduction

Granular computing (GrC) is a label of theories, methodologies, techniques, and tools that make use of granules, i.e., groups, classes, or clusters of a universe, in the process of problem solving [15, 20]. There is a fast growing and renewed interest in the study of granular computing [4, 8].

There are many fundamental issues in granular computing, such as granulation of the universe, description of granules, relationships between granules, and computing with granules [15]. Many models of granular computing have been proposed and studied [16, 20]. However, each model only captures certain aspects of granular computing. There is still a need for formal and concrete models for systematic studies of fundamental issues of granular computing.

The main objective of this paper is to propose a concrete model of granular computing based on a simple granulation structure, namely, a partition of a universe. Results from rough sets [9], quotient space theory [21, 22], belief functions [12], and power algebra [1] are reformulated, re-interpreted, and combined for granular computing. With respect a two-level granulation structure, namely, a universe and the quotient universe induced by an equivalence relation, we introduce two basic operations called zooming-in and zooming-out operations. Zooming-in allows us to expand an element of the quotient universe into a subset of the universe, and hence reveals more detailed information. Zooming-out allows us to move to the quotient universe by ignoring some details. Computations in both universes can be connected through zooming operations. The study of two-level structures is generalized to multi-level granulation structures.

2 Granulation and Approximations

This section studies a simple granulation of a universe, i.e., a partition induced by an equivalence relation. The discussion is basically a reformulation, re-interpretation, and combination of results from Shafer's work on belief functions [12], Pawlak's work on rough sets [9], and Zhang and Zhang's work on quotient space theory for problem solving [21, 22].

2.1 Granulation by an equivalence relation

Suppose U is a finite and nonempty set called the universe. Let $E \subseteq U \times U$ be an equivalence relation on U. The pair $apr = (U, E)$ is called an approximation space [9, 10]. The relation E can be conveniently represented by a mapping from U to 2^U, where 2^U is the power set of U. The mapping $[\cdot]_E : U \longrightarrow 2^U$ is given by: $[x]_E = \{y \in U \mid xEy\}$. The subset $[x]_E$ is the equivalence class containing x. The family of all equivalence classes is commonly known as the quotient set and is denoted by $U/E = \{[x]_E \mid x \in U\}$. It defines a partition of the universe, namely, a family of pairwise disjoint subsets whose union is the universe.

Under the equivalence relation, we only have a coarsened view of the universe. Each equivalence class is considered as a whole granule instead of many individuals [16]. They are considered as the basic or elementary definable, observable, or measurable subsets of the universe [10, 14]. The equivalence class $[x]_E$ containing x plays dual roles. It is a subset of U if considered in relation to the universe, and an element of U/E if considered in relation to the quotient set. In clustering, one typically associates a name with a cluster such that elements of the cluster are instances of the named category or concept [5]. Lin [7], following Dubois and Prade [3], explicitly used $[x]_E$ for representing a subset of U and $Name([x]_E)$ for representing an element of U/E.

With a partition, we have two views of the same universe, a coarse-grained view U/E and a detailed view U. A concept, represented as a subset of a universe, is thus described differently under different views. As we move from one view to the other, we change our perceptions and representations of the same concept.

2.2 Zooming-in: refinement

Each element in the coarse-grained universe U/E is a name associated with a subset of the universe U. For an element $X_i \in U/E$, a detailed view can be obtained at the level of U from the subset of U corresponding to X_i. This can be easily extended to any subset $X \subseteq U/E$. The expansion of an element of U/E into a subset of U is referred to as a zooming-in operation. With zooming-in, we are able to see more details.

Formally, zooming-in can be defined by an operator $\omega : 2^{U/E} \longrightarrow 2^U$. Shafer [12] referred to the zooming-in operation as refining. For a singleton subset $\{X_i\} \in 2^{U/E}$, we define [3]:

$$\omega(\{X_i\}) = [x]_E, \quad X_i \text{ is the name of the equivalence class } [x]_E. \tag{1}$$

For an arbitrary subset $X \subseteq U/E$, we have:

$$\omega(X) = \bigcup_{X_i \in X} \omega(\{X_i\}). \tag{2}$$

By zooming-in on a subset $X \subseteq U/E$, we obtain a unique subset $\omega(X) \subseteq U$. The set $\omega(X) \subseteq U$ is called the refinement of X.

The zooming-in operation has the following properties [12]:

$$\begin{array}{ll} \text{(zi1)} & \omega(\emptyset) = \emptyset, \\ \text{(zi2)} & \omega(U/E) = U, \\ \text{(zi3)} & \omega(X^c) = (\omega(X))^c, \\ \text{(zi4)} & \omega(X \cap Y) = \omega(X) \cap \omega(Y), \\ \text{(zi5)} & \omega(X \cup Y) = \omega(X) \cup \omega(Y), \\ \text{(zi6)} & X \subseteq Y \Longleftrightarrow \omega(X) \subseteq \omega(Y), \end{array}$$

where ^{c} denotes the set complement operator, the set-theoretic operators on the left hand side apply to the elements of $2^{U/E}$, and the same operators on the right hand side apply to the elements of 2^U. From these properties, it can be seen that any relationships of subsets observed under coarse-grained view would hold under the detailed view, and vice versa. For example, in addition to (zi6), we have $X \cap Y = \emptyset$ if and only if $\omega(X) \cap \omega(Y) = \emptyset$, and $X \cup Y = U/E$ if and only if $\omega(X) \cup \omega(Y) = U$. Therefore, conclusions drawn based on the coarse-grained elements in U/E can be carried over to the universe U.

2.3 Zooming-out: approximations

The change of views from U to U/E is called a zooming-out operation. By zooming-out, a subset of the universe is considered as a whole rather than many individuals. This leads to a loss of information. Zooming-out on a subset $A \subseteq U$ may induce an inexact representation in the coarse-grained universe U/E.

The theory of rough sets focuses on the zooming-out operation. For a subset $A \subseteq U$, we have a pair of lower and upper approximations in the coarse-grained universe [2, 3, 13]:

$$\begin{aligned} \underline{apr}(A) &= \{[x]_E \mid [x]_E \in U/E, [x]_E \subseteq A\}, \\ \overline{apr}(A) &= \{[x]_E \mid [x]_E \in U/E, [x]_E \cap A \neq \emptyset\}. \end{aligned} \tag{3}$$

The expression of lower and upper approximations as subsets of U/E, rather than subsets of U, has only been considered by a few researchers in rough set community [2, 3, 13, 19]. On the other hand, such notions have been considered in other contexts. Shafer [12] introduced those notions in the study of belief functions and called them the inner and outer reductions of $A \subseteq U$ in U/E. The connections between notions introduced by Pawlak in rough set theory and these introduced by Shafer in belief function theory have been pointed out by Dubois and Prade [3].

The expression of approximations in terms of elements of U/E clearly shows that representation of A in the coarse-grained universe U/E. By zooming-out, we only obtain an approximate representation. The lower and upper approximations satisfy the following properties [12, 19]:

$$
\begin{array}{ll}
\text{(zo1)} & \underline{apr}(\emptyset) = \overline{apr}(\emptyset) = \emptyset, \\
\text{(zo2)} & \underline{apr}(U) = \overline{apr}(U) = U/E, \\
\text{(zo3)} & \underline{apr}(A) = (\overline{apr}(A^c))^c, \qquad \overline{apr}(A) = (\underline{apr}(A^c))^c; \\
\text{(zo4)} & \underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B), \qquad \overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B), \\
\text{(zo5)} & \underline{apr}(A) \cup \underline{apr}(B) \subseteq \underline{apr}(A \cup B), \qquad \overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B), \\
\text{(zo6)} & A \subseteq B \Longrightarrow [\underline{apr}(A) \subseteq \underline{apr}(B), \overline{apr}(A) \subseteq \overline{apr}(B)], \\
\text{(zo7)} & \underline{apr}(A) \subseteq \overline{apr}(A).
\end{array}
$$

According to properties (zo4)-(zo6), relationships between subsets of U may not be carried over to U/E through the zooming-out operation. It may happen that $A \cap B \neq \emptyset$, but $\underline{apr}(A \cap B) = \emptyset$, or $A \cap B = \emptyset$, but $\overline{apr}(A \cap B) \neq \emptyset$. Similarly, we may have $A \neq B$, but $\underline{apr}(A) = \underline{apr}(B)$ and $\overline{apr}(A) = \overline{apr}(B)$. Nevertheless, we can draw the following inferences:

$$
\begin{array}{ll}
\text{(i1)} & \underline{apr}(A) \cap \underline{apr}(B) \neq \emptyset \Longrightarrow A \cap B \neq \emptyset, \\
\text{(i2)} & \overline{apr}(A) \cap \overline{apr}(B) = \emptyset \Longrightarrow A \cap B = \emptyset, \\
\text{(i3)} & \underline{apr}(A) \cup \underline{apr}(B) = U/E \Longrightarrow A \cup B = U, \\
\text{(i4)} & \overline{apr}(A) \cup \overline{apr}(B) \neq U/E \Longrightarrow A \cup B \neq U.
\end{array}
$$

If $\underline{apr}(A) \cap \underline{apr}(B) \neq \emptyset$, by property (zo4) we know that $\overline{apr}(A) \cap \overline{apr}(B) \neq \emptyset$. We say that A and B have a non-empty overlap, and hence are related, in U/E. By (i1), A and B must have a non-empty overlap, and hence are related, in U. Similar explanations can be associated with other inference rules.

2.4 Zooming-out and zooming-in: classical rough set approximations

Given a subset $A \subseteq U$, we perform the zooming-out operation and obtain a pair of subsets $\underline{apr}(A) \subseteq \overline{apr}(A) \subseteq U/E$. By zooming-in on the two subsets, we derive the classical rough set approximations [9, 10, 13]:

$$
\begin{aligned}
\omega(\underline{apr}(A)) &= \bigcup_{X_i \in \underline{apr}(A)} \omega(\{X_i\}) = \bigcup \{[x]_E \mid [x]_E \in U/E, [x]_E \subseteq A\}, \\
\omega(\overline{apr}(A)) &= \bigcup_{X_i \in \overline{apr}(A)} \omega(\{X_i\}) = \bigcup \{[x]_E \mid [x]_E \in U/E, [x]_E \cap A \neq \emptyset\}. \quad (4)
\end{aligned}
$$

For a subset $X \subseteq U/E$ we can zoom-in and obtain a subset $\omega(X) \subseteq U$, and then zoom-out to obtain a pair of subsets $\underline{apr}(\omega(X))$ and $\overline{apr}(\omega(X))$. The compositions of zooming-in and zooming-out operations have the properties [12]: for

$X \subseteq U/E$ and $A \subseteq U$,

$$\begin{array}{ll} \text{(zio1)} & \omega(\underline{apr}(A)) \subseteq A \subseteq \omega(\overline{apr}(A)), \\ \text{(zio2)} & \underline{apr}(\omega(X)) = \overline{apr}(\omega(X)) = X. \end{array}$$

The composition of zooming-out and zooming-in cannot recover the original set $A \subseteq U$. The composition zooming-in and zooming-out produces the original set $X \subseteq U/E$. A connection between the zooming-in and zooming-out operations can be established. For a pair of subsets $X \subseteq U/E$ and $A \subseteq U$, we have [12]:

$$\begin{array}{ll} (1) & w(X) \subseteq A \Longleftrightarrow X \subseteq \underline{apr}(A), \\ (2) & A \subseteq \omega(X) \Longleftrightarrow \overline{apr}(A) \subseteq X. \end{array}$$

Property (1) can be understood as follows. Any subset $X \subseteq U/E$, whose refinement is a subset of A, is a subset of the lower approximation of A. Only a subset of the lower approximation of A has a refinement that is a subset of A. It follows that $\underline{apr}(A)$ is the largest subset of U/E whose refinement is contained in A, and $\overline{apr}(A)$ is the smallest subset of U/E whose refinement containing A.

The rough set approximations can be related to the notion of dilation and erosion operations in mathematical morphology [11]. They have been considered by Zhang and Zhang in the quotient space theory for problem solving [21, 22].

3 Granulation and Computations

The zooming-in and zooming-out operations provide a linkage between subsets of universe and subsets of granulated universe. Based on such connections, we can study the relationships between computations in two different universes.

3.1 Zooming-in: set-based computations

Suppose $f : U \longrightarrow \Re$ is a real-valued function on U. One can lift the function f to U/E by performing set-based computations [17]. The lifted function f^+ is a set-valued function that maps an element of U/E to a subset of real numbers. More specifically, for an element $X_i \in U/E$, the value of function is given by:

$$f^+(X_i) = \{f(x) \mid x \in \omega(\{X_i\})\}. \tag{5}$$

The function f^+ can be changed into a single-valued function f_0^+ in a number of ways. For example, Zhang and Zhang [21] suggested the following methods:

$$\begin{aligned} f_0^+(X_i) &= \min f^+(X_i) = \min\{f(x) \mid x \in \omega(\{X_i\})\}, \\ f_0^+(X_i) &= \max f^+(X_i) = \max\{f(x) \mid x \in \omega(\{X_i\})\}, \\ f_0^+(X_i) &= \text{average} f^+(X_i) = \text{average}\{f(x) \mid x \in \omega(\{X_i\})\}. \end{aligned} \tag{6}$$

The minimum, maximum, and average definitions may be regarded as the most permissive, the most optimistic, and the balanced view in moving functions from U to U/E. More methods can be found in the book by Zhang and Zhang [21].

For a binary operation $\circ$ on U, a binary operation $\circ^+$ on U/E is defined by [1, 17]:

$$X_i \circ^+ X_j = \{x_i \circ x_j \mid x_i \in \omega(\{X_i\}), x_j \in \omega(\{X_j\})\}, \tag{7}$$

In general, one may lift any operation p on U to an operation p^+ on U/E, called the power operation of p. Suppose $p : U^n \longrightarrow U$ $(n \geq 1)$ is an n-ary operation on U. Its power operation $p^+ : (U/E)^n \longrightarrow 2^U$ is defined by [1]:

$$p^+(X_0, \ldots, X_{n-1}) = \{p(x_0, \ldots, x_{n-1}) \mid x_i \in \omega(\{X_i\}) \text{ for } i = 0, \ldots, n-1\}, \tag{8}$$

for any $X_0, \ldots, X_{n-1} \in U/E$. This provides a universal-algebraic construction approach. For any algebra $(U, p_1, \ldots, p_k)$ with base set U and operations $p_1, \ldots, p_k$, its quotient algebra is given by $(U/E, p_1^+, \ldots, p_k^+)$.

The power operation p^+ may carry some properties of p. For example, for a binary operation $p : U^2 \longrightarrow U$, if p is commutative and associative, p^+ is commutative and associative, respectively. If e is an identity for some operation p, the set $\{e\}$ is an identity for p^+. Many properties of p are not carried over by p^+. For instance, if a binary operation p is idempotent, i.e., $p(x, x) = x$, p^+ may not be idempotent. If a binary operation g is distributive over p, g^+ may not be distributive over p^+.

3.2 Zooming-out: inner and outer fuzzy measures

Suppose $\mu : 2^{U/E} \longrightarrow [0, 1]$ is a set function on U/E. If μ satisfies the conditions, (i). $\mu(\emptyset) = 0$, (ii). $\mu(U/E) = 1$, and (iii). $X \subseteq Y \Longrightarrow \mu(X) \leq \mu(Y)$, μ is called a fuzzy measure [6]. Examples of fuzzy measures are probability functions, possibility and necessity functions, and belief and plausibility functions.

Information about subsets in U can be obtained from μ on U/E and the zooming-out operation. For a subset $A \subseteq U$, we can define a pair of inner and outer fuzzy measures [18]:

$$\underline{\mu}(A) = \mu(\underline{apr}(A)), \qquad \overline{\mu}(A) = \mu(\overline{apr}(A)). \tag{9}$$

They are fuzzy measures. If μ is a probability function, $\underline{\mu}$ and $\overline{\mu}$ are a pair of belief and plausibility functions [12, 18]. If μ is a belief function, $\underline{\mu}$ is a belief function, and if μ is a plausibility function, $\overline{\mu}$ is a plausibility [18].

4 Multi-level Granulations and Approximations

The simple granulation structure can be used to construct a multi-level granulation or a hierarchy [16, 21]. The zooming-in and zooming-out operations can be defined on any two adjacent levels.

The ground level, l_0, is the universe $U_0 = U$. The next level, l_1, constructed through an equivalence relation E_0 or the corresponding partition, is $U_1 = U/E_0$. The elements of U_1 are names assigned to subsets of U_0. From the universe U_i at level $i \geq 0$, the next level of universe is given by $U_{i+1} = U_i/E_i$, where E_i is an equivalence relation on U_i. The elements of U_{i+1} are names associated with subsets of U_i. Thus, we have a multi-level representation of the universe U with different grain sizes [21].

The connection between different granulated views can be easily established by the zooming-in and zooming-out operations. Let ω_i and $(\underline{apr}_i, \overline{apr}_i)$, $i \geq 0$, denote the zooming-in and zooming-out operations defined between U_{i+1} and U_i. For a ground level subset A, its approximations at level $k > 0$ is given by:

$$\underline{apr}_{k-1}(\ldots \underline{apr}_0(A)), \qquad \overline{apr}_{k-1}(\ldots \overline{apr}_0(A)). \tag{10}$$

They are obtained by the composition of k zooming-out operations, namely, $\underline{apr}_{k-1} \circ \ldots \circ \underline{apr}_0$, and $\overline{apr}_{k-1} \circ \ldots \circ \overline{apr}_0$. For a subset $X \subseteq U_k$, its ground level refinement is given by:

$$\omega_0(\ldots \omega_{k-1}(X)). \tag{11}$$

It is obtained by the composition of k zooming-in operations, namely, $\omega_0 \circ \ldots \circ \omega_{k-1}$. We can establish a connection between subsets in any two levels in the multi-level granulation by treating the lowest level as the ground level.

By zooming-in, we can move downwards in the multi-level granulation, which enables us to examine more details. By zooming-out, we can move upwards in the multi-level granulation, which enables us to ignore details and obtain a more abstract level description. Zhang and Zhang [21] applied this simple principle to solve many problems in artificial intelligence and developed a granular computing theory for problem solving.

Although we illustrate the idea of granulations, as well as the zooming-in and zooming-out operations, by using a simple and concrete framework based on the notion of partitions, the argument can be easily applied to other granulation structures. For example, one may use partial orders to model granulation structures [21]. A crucial step is the introduction of operations that allow us to navigate (change views) in the granulation structures. The operations, such as zooming-in and zooming-out, also allow us to establish connections between elements of different granulated views of the universe. It may be desirable that the granulated universe keep some properties of the original universe [21]. Computations under multi-level granulation structures can be done by extending the computational methods presented in Section 3 under the two-level structure.

5 Conclusion

We review, compare, reformulate and re-interpret results from rough sets, quotient space theory, belief functions, and power algebras. The results enable us to build a concrete model for granular computing. The granular computing model offers a unified view to examine many notions studied in several related fields. More insights may be obtained in one particular field from studies in other fields. The combination of results from different fields under the umbrella term of granular computing may eventually lead to new theories and methodologies for problem solving.

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