Granular Computing using Neighborhood Systems

Y.Y. Yao

Department of Computer Science
University of Regina
Regina, Saskatchewan, Canada S4S 0A2
Email: yyao@cs.uregina.ca

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Abstract

A set-theoretic framework is proposed for granular computing. Each element of a universe is associated with a nonempty family of neighborhoods. A neighborhood of an element consists of those elements that are drawn towards that element by indistinguishability, similarity, proximity, or functionality. It is a granule containing the element. A neighborhood system is a family of granules, which is the available information or knowledge for granular computing. Operations on neighborhood systems, such as complement, intersection, and union, are defined by extending set-theoretic operations. They provide a basis of the proposed framework of granular computing. Using this framework, we examine the notions of rough sets and qualitative fuzzy sets.

1 Introduction

In many situations, it is impossible or unnecessary to distinguish individual objects or elements in a universe. For example, if a group of patients are described by using several symptoms, many patients would share the same symptoms, and hence are indistinguishable. This forces us to think a subset of the patients as one unit, instead of many individuals. In other words, one has to consider groups, classes, or clusters of elements. They are referred to as granules and can be either crisp or fuzzy. Elements in a granule may be drawn together by indistinguishability, similarity, proximity, or functionality [24]. Such a clustering of elements leads to information or knowledge granulation, which forms a basis of granular computing. There are at least two reasons for the study of granular computing [25, 26]. When a problem involves incomplete, uncertain, or vague information, it may be difficult to differentiate distinct elements and one is forced to consider granules [16]. Alternatively, although detailed information may be available, it may be sufficient to use granules in order to have an efficient and practical solution. Very precise solutions may in fact not be required for many practical problems.

The main objective of this paper is to propose a set-theoretic framework for granular computing using neighborhood systems [10, 21]. In this framework, an element of a universe is associated with a nonempty family of subsets of the universe. The family is called a neighborhood system of the element, and each subset is called a neighborhood of the element. Each neighborhood of an element may be regarded as a granule containing that element, and a neighborhood system of the element is a family of granules. Operations on neighborhood systems, such as complement, intersection, and union, will be defined by extending standard set-theoretic operations [21]. It is done in the manner used by Brink [3] for the development of power algebras. An ordering relation can also be defined from a neighborhood system. These notions establish a basis of proposed framework for granular computing.

To demonstrate the usefulness of the proposed framework, two problems will be discussed. The first problem deals with concept formulation and approximation. The solution is related to rough set approach [15]. The second problem is related to the interpretation of fuzzy sets [23]. A crisp set is represented by 1-neighborhood systems, i.e., each element has exactly one neighborhood. Set-theoretic operations are interpreted in terms of neighborhood system operations. Following the same argument, one may represent a fuzzy set by a nested neighborhood system. Each neighborhood corresponds to a particular α-cut of the fuzzy set. The notion of qualitative fuzzy sets is introduced. Such a notion may provide further insights to the understanding of fuzzy set membership functions and fuzzy set-theoretic operations. In summary, the proposed framework may provide a unified model for the study of two well known granular computing theories. It should also be pointed out that in the present study we only consider the set-theoretic facet of rough sets and fuzzy sets.

2 Overview of Neighborhood Systems

The concept of neighborhood systems is originated from studies of topological space and its generalization called Frechet (V)space [17]. Recently, Lin [9, 10] adopted this notion as a tool to describe relationships between objects.

2.1 Basic concepts

For an element x of a finite universe U, one associates with it a subset $n(x) \subseteq U$ called the *neighborhood* of x. Intuitively speaking, elements in a neighborhood of an element are somewhat indiscernible or at least not noticeably distinguishable to x. That is, elements in a neighborhood of x are drawns towards x by indistinguishability, similarity, or functionality [11]. In general, a neighborhood of x may or may not contain x. A neighborhood of x containing x is called a reflexive neighborhood. For the present study, we are only interested in reflexive neighborhoods of x, in order to accommodate the previous intuitive interpretation of neighborhoods. A neighborhood system $\mathrm{NS}(x)$ of x is a nonempty family of neighborhoods of x. A neighborhood system $\mathrm{NS}(x)$ of x groups the universe into classes. Distinct neighborhoods x consist of elements having different types of, or various degrees of, similarity to x. A neighborhood system may be formally interpreted as an operator from U to 2^{2^U} that maps each element to a family of subsets of the universe. Let $\mathrm{NS}(U)$ denote the collection of neighborhood systems for all elements in U. It determines a Frechet (V)space, written $(U, \mathrm{NS}(U))$. There is no additional requirements on neighborhood systems.

Example 1 Consider a universe $U = \{a, b, c, d\}$. We can have the following neighborhood systems for elements of U:

$$\begin{aligned}
\mathrm{NS}(a) &= \{\{a\}, \{a, b\}\}, \\
\mathrm{NS}(b) &= \{\{a, b\}, \{a, b, c\}\}, \\
\mathrm{NS}(c) &= \{\{c\}\}, \\
\mathrm{NS}(d) &= \{\{d\}, \{b, d\}, \{a, b, d\}\}.
\end{aligned}$$

All neighborhoods are reflexive. For elements a and d, a is in a neighborhood of d, but d is not in any neighborhood of a. This implies that neighborhood relationships may not be symmetric. Consider the neighborhood system of a. We may say that two types of similarity are represented: one suggests that a is only similar to itself, the other suggests that b is similar to a. The former may be considered to be a strong similarity and the latter a weak similarity.

The notion of neighborhood systems provides a convenient tool for representing relationship between objects in a universe. As shown by the following examples, it can be used to represent both quantitative and qualitative similarities.

Example 2 Consider a universe U and a distance function $D : U \times U \longrightarrow \mathcal{R}^+$, where $\mathcal{R}^+$ is the set of non-negative real numbers. For each number $d \in \mathcal{R}^+$, we may define the following neighborhood of $x \in U$:

$$n_d(x) = \{y \mid D(x, y) \leq d\}.$$

A neighborhood system of x is given by $\mathrm{NS}(x) = \{n_d(x) \mid d \in \mathcal{R}^+\}$. Conversely, we have $D(x, y) = \sup\{d \mid y \in n_d(x)\}$. That is, we may interpret quantitative similarity (expressed using a distance function) in terms of neighborhood systems. The use of a quantitative distance function implies that any two elements can be compared regarding their similarity to a particular element of U. In terms of neighborhood, we have $n_a(x) \subseteq n_b(x)$ for any two numbers $a, b \in \mathcal{R}^+$ with $a \geq b$. The neighborhood system $\mathrm{NS}(x)$ is a nested family of subsets of U, with each neighborhood representing a specific level of similarity to x.

Example 3 Suppose $R \subseteq U \times U$ is a binary relation on a universe U. Using R, for each element $x \in U$, we define four neighborhoods [20]:

$$\begin{aligned} R_s(x) &= \{y \mid xRy\}, \\ R_p(x) &= \{y \mid yRx\}, \\ R_{p\wedge s}(x) &= \{y \mid xRy \text{ and } yRx\}, \\ R_{p\vee s}(x) &= \{y \mid xRy \text{ or } yRx\}. \end{aligned}$$

They are referred to as the successor, predecessor, successor-and-predecessor, and successor-or-predecessor neighborhoods of x. The neighborhood system of x is given by $\mathrm{NS}(x) = \{R_s(x), R_p(x), R_{p\wedge s}(x), R_{p\vee s}(x)\}$. If R is a reflexive relation, all these neighborhoods are reflexive. If R is a symmetric relation, all neighborhoods become the same one. Obviously, unlike the distance function based neighborhood systems, two elements of U may not be comparable.

The representation of qualitative similarity using neighborhood systems may be more useful in some applications, where the meaning of a distance or a similarity function is not clear. It may also be case that the qualitative information, i.e., the order implied by the numeric values, is useful rather than the precise numeric values.

2.2 Operations on neighborhood systems

Let U be a set and $\circ$ a binary operation on U. One can define a binary operation $\circ^+$ on subsets of U as follows [3]:

$$X \circ^+ Y = \{x \circ y \mid x \in X, y \in Y\}, \tag{1}$$

for any $X, Y \subseteq U$. In general, one may lift any operation f on elements of U to an operation f^+ on subsets of U, called the power operation of f. Suppose $f : U^n \longrightarrow U$ $(n \geq 1)$ is an n-ary operation on U. The power operation $f^+ : (2^U)^n \longrightarrow 2^U$ is defined by [3]:

$$f^+(X_0, \ldots, X_{n-1}) = \{f(x_0, \ldots, x_{n-1}) \mid x_i \in X_i \text{ for } i = 0, \ldots, n-1\}, \tag{2}$$

for any $X_0, \ldots, X_{n-1} \subseteq U$. This provides a universal-algebraic construction approach. For any algebra $(U, f_1, \ldots, f_k)$ with the base set U and operations $f_1, \ldots, f_k$, its power algebra is given by $(2^U, f_1^+, \ldots, f_k^+)$. The power operation f^+ may carry some properties of f. For example, for a binary operation $f : U^2 \longrightarrow U$, if f is commutative and associative, f^+ is commutative and associative, respectively. If e is an identity for some operation f, the set $\{e\}$ is an identity for f^+. If an unary operation $f : U \longrightarrow U$ is an involution, i.e., $f(f(x)) = f(x)$, f^+ is also an involution. On the other hand, many properties of f are not carried over by f^+. For instance, if a binary operation f is idempotent, i.e., $f(x, x) = x$, f^+ may not be idempotent. If a binary operation g is distributive over f, g^+ may not be distributive over f^+.

A neighborhood system is a family of subsets of the universe U. By applying the idea of power algebras, we may lift set-theoretic operations on sets to neighborhood systems. For two neighborhood systems NS_1 and NS_2, the complement, intersection and union are defined by:

$$\begin{aligned} \neg \mathrm{NS}_1(x) &= \{\sim n_i(x) \mid n_i(x) \in \mathrm{NS}_1(x)\}, \\ \mathrm{NS}_1(x) \sqcap \mathrm{NS}_2(x) &= \{n_i(x) \cap n_j(x) \mid n_i(x) \in \mathrm{NS}_1(x), n_j(x) \in \mathrm{NS}_2(x)\}, \\ \mathrm{NS}_1(x) \sqcup \mathrm{NS}_2(x) &= \{n_i(x) \cup n_j(x) \mid n_i(x) \in \mathrm{NS}_1(x), n_j(x) \in \mathrm{NS}_2(x)\}. \end{aligned}$$

These operations can be interpreted as extensions of set-theoretic operations in a framework of set-based computations [22]. Different neighborhood systems may be provided by different experts or defined based on different features of objects. The above operations may be used for combining neighborhood systems.

It can be easily verified that the previous set-theoretic neighborhood system operations have the following properties:

$$\begin{aligned} \text{Commutativity}: &\quad \mathrm{NS}_1(x) \sqcap \mathrm{NS}_2(x) = \mathrm{NS}_2(x) \sqcap \mathrm{NS}_1(x), \\ &\quad \mathrm{NS}_1(x) \sqcup \mathrm{NS}_2(x) = \mathrm{NS}_2(x) \sqcup \mathrm{NS}_1(x); \\ \text{Associativity}: &\quad (\mathrm{NS}_1(x) \sqcap \mathrm{NS}_2(x)) \sqcap \mathrm{NS}_3(x) = \\ &\quad \mathrm{NS}_1(x) \sqcap (\mathrm{NS}_2(x) \sqcap \mathrm{NS}_3(x)), \\ &\quad (\mathrm{NS}_1(x) \sqcup \mathrm{NS}_2(x)) \sqcup \mathrm{NS}_3(x) = \\ &\quad \mathrm{NS}_1(x) \sqcup (\mathrm{NS}_2(x) \sqcup \mathrm{NS}_3(x)); \\ \text{De Morgan's law}: &\quad \neg(\mathrm{NS}_1(x) \sqcap \mathrm{NS}_2(x)) = \neg\mathrm{NS}_1(x) \sqcup \neg\mathrm{NS}_2(x), \\ &\quad \neg(\mathrm{NS}_1(x) \sqcup \mathrm{NS}_2(x)) = \neg\mathrm{NS}_1(x) \sqcap \neg\mathrm{NS}_2(x); \\ \text{Double negation law}: &\quad \neg\neg\mathrm{NS}_1(x) = \mathrm{NS}_1(x); \\ \text{Unit element of } \sqcap: &\quad \mathrm{NS}_1(x) \sqcap \{U\} = \mathrm{NS}_1(x); \\ \text{Zero element of } \sqcup: &\quad \mathrm{NS}_1(x) \sqcup \{\emptyset\} = \mathrm{NS}_1(x). \end{aligned}$$

Operators $\sqcap$ and $\sqcup$ are not idempotent. Each of them is not distributive over the other. In general, $\mathrm{NS}(x) \sqcap \neg\mathrm{NS}(x)$ is not equal to $\{\emptyset\}$, and $\mathrm{NS}(x) \sqcup \neg\mathrm{NS}(x)$ is not equal to $\{U\}$. A detailed study of such a system can be found in a paper by Brink [2] on second-order Boolean algebras.

2.3 Orderings induced by neighborhood systems

In the last two subsections, the notion of neighborhood systems is formulated mainly from a purely mathematical point of view. A semantic interpretation is necessary, if we want to apply the theory to practical problems. In what follows, we interpret a neighborhood system by the more familiar notion of an ordering characterized by a "closer to" relation.

A neighborhood system models nearness between two elements in a space. It is originated from the abstraction of geometric notion, in which two points are "close to" or "approximate to" each other [11]. Each neighborhood of an element x of a universe U is a set of elements from U that are drawn towards x by their similarity to x. To make this interpretation more precise, we rely on the following intuitive results. An element must be close to itself. That is, all neighborhoods must be reflexive. For two neighborhoods with $n_i(x) \subset n_j(x)$, elements in $n_i(x)$ are closer to x than elements in $n_j(x) - n_i(x)$. This seems to be reasonable, especially for the nested neighborhood systems given in Example 2. By extending the argument to two arbitrary neighborhoods $n_i(x)$ and $n_j(x)$, one may say that elements in both neighborhoods are closer to x than those in only one neighborhood. That is, elements in $n_i(x) \cap n_j(x)$ are closer to x than elements in $n_i(x) \cup n_j(x) - n_i(x) \cap n_j(x)$. The same argument may be applied to a neighborhood system $\mathrm{NS}(x)$.

For a neighborhood system $\mathrm{NS}(x)$, the $\cap$-closure $\mathrm{NS}^*(x)$ is defined to be the minimum subset of 2^U, which contains $\mathrm{NS}(x)$ and the entire set U, and is closed under intersection $\cap$. The $\cap$-closure of a neighborhood system is a complete lattice under the ordering given by set inclusion. The meet is given by set intersection, but the join is different from the set union. It is commonly referred to as a closure system [4]. Based on the $\cap$-closure $\mathrm{NS}^*(x)$ of a neighborhood system $\mathrm{NS}(x)$, a binary relation $\prec$ on U is defined as follows:

$$\begin{aligned} a \prec b \iff & \text{ there exists } n_1(x), n_2(x) \in \mathrm{NS}^*(x) \text{ such that } n_1(x) \subset n_2(x), \\ & a \in n_1(x), \text{ and } b \in n_2(x) - \bigcup_{n(x) \subset n_2(x)} n(x). \end{aligned} \tag{3}$$

For two elements $a, b \in U$ with $a \prec b$, a is said to be closer to x than b. The relation $\prec$ shows the structure imposed by a neighborhood system. It can be verified that the relation $\prec$ is asymmetric and transitive. The binary relation defined by:

$$a \preceq b \iff a \prec b \text{ or } a = b, \tag{4}$$

is a partial order. In fact, a diagram for relation $\prec$ can be easily obtained from the Hasse diagram of the lattice $\mathrm{NS}^*(x)$: for a subset $n(x) \in \mathrm{NS}^*(x)$, deleting all elements in subsets in $\mathrm{NS}^*(x)$ proceeding $n(x)$.

Example 4 A nested neighborhood system of an element $x \in U$ is a family of subsets of U, $\mathrm{NS}(x) = \{n_1(x), \ldots, n_k(x)\}$, such that

$$n_1(x) \subset n_2(x) \subset \ldots \subset n_k(x).$$

The $\cap$-closure of $\mathrm{NS}(x)$ is itself if $n_k(x) = U$, otherwise it is $\mathrm{NS}(x) \cup \{U\}$. The following binary relation indicates the closeness of elements of U to x:

$$n_1(x) \prec n_2(x) - n_1(x) \prec \ldots \prec n_k(x) - n_{k-1}(x) \prec U - n_k(x).$$

In the above representation, we have implicitly extended the relation $\prec$ to subsets of the universe.

Example 5 Consider a universe $U = \{a, b, c, d, e, f\}$ and a neighborhood system of a defined by:

$$\mathrm{NS}(a) = \{\{a, e\}, \{a, b, c\}, \{a, b, d\}\}.$$

The $\cap$-closure of $\mathrm{NS}(a)$ is given by:

$$\mathrm{NS}^*(a) = \{\{a\}, \{a, b\}, \{a, e\}, \{a, b, c\}, \{a, b, d\}, U\}.$$

The lattice $\mathrm{NS}^*(a)$ and the induced relation $\prec$ are given by:

$$\{a\} \subseteq \begin{array}{c} \{a,b\} \subseteq \begin{array}{c}\{a,b,c\} \\ \{a,b,d\}\end{array} \\ \{a,e\} \end{array} \subseteq U \qquad \text{and} \qquad a \prec \begin{array}{c} b \prec \begin{array}{c} c \\ d \end{array} \\ e \end{array} \prec f.$$

From this example, we can see that e is not comparable with b, c, or d regarding their closeness to a.

3 Neighborhood Systems and Granular Computing

Based on the notion of neighborhood systems, we consider in this section problems of concept formation and representation. Our discussion centers on the understanding that each subset of a universe is a granule representing a certain concept. With specific interpretations of sets of granules, i.e., families of subsets of the universe, we may approximate a precise concept or represent a vague concept. The former is related to rough sets, and the latter is related to qualitative fuzzy sets.

3.1 Concepts and granules

In the study of formal concepts, every concept is understood as a unit of thoughts that consists of two parts, the extension and intension of the concept [14, 18]. The extension of a concept is the set of objects or entities which are instances of the concept. All objects in the extension have the same properties that characterize the concept. The intension (comprehension) of a concept consists of all properties or attributes that are valid for all those objects to which the concept applies. A concept is thus described jointly by its extension and intension, i.e., a set of objects and a set of properties. This formulation enables us to study formal concepts in a set-theoretic framework.

Consider a universe U. Each subset $A \subseteq U$ may be interpreted as the extension of certain concept. All elements of A are characterized by certain properties, namely, the intension of the concepts. In the following discussion, we only consider the extension of a concepts without explicitly referring to its intension. One may say that elements in A are similar or indistinguishable based solely on properties in the intension. That is, we regard the elements of A to be equivalent, as they are all instances of the same concept. Under this interpretation, we may characterize a concept trivially by a family of 1-neighborhood systems. With respect to the concept A, we define:

$$\mathrm{NS}_A(x) = \begin{cases} \{A\} & x \in A, \\ \\ \{\emptyset\} & x \notin A. \end{cases} \tag{5}$$

That is, the set A is the neighborhood of every element of A, and the empty set $\emptyset$ is the neighborhood of all elements not in A. It is straightforward to verify that set-theoretic operations can be expressed as neighborhood system operations using such a representation. One may treat A as being a granule in the sense that elements in A are drawn together by their similarities, i.e., they are all instances of a certain concept. From the above discussion, we may use "neighborhoods", "concepts", "subsets", and "granules" interchangeably. The choice of a particular name is not as important as the intended meaning associated with each subset of the universe.

3.2 Rough set approximations

Let $E \subseteq U \times U$ be an equivalence relation on a finite and nonempty universe U. That is, the relation E is reflexive, symmetric, and transitive. The pair $apr = (U, E)$ is called a Pawlak approximation space. The equivalence relation E partitions the universe U into disjoint subsets called equivalence classes. Elements in the same equivalence class are said to be indistinguishable. Equivalence classes of E are called elementary sets. Let

$$[x]_E = \{y \mid xEy\}, \tag{6}$$

denote the equivalence class containing x. It may be interpreted as a granule containing x. If a precise concept $A \subseteq U$ is represented relative to granules $[x]_E$'s, it is no longer precise. A subset $A \subseteq U$ is approximated by a pair of sets as follows [20]:

$$\begin{aligned} \underline{apr}(A) &= \bigcup \{[x]_E \mid x \in U, [x]_E \subseteq A\}, \\ \overline{apr}(A) &= \bigcup \{[x]_E \mid x \in U, [x]_E \cap A \neq \emptyset\}. \end{aligned} \tag{7}$$

The lower approximation of A is the union of equivalence classes that are subsets of A, and the upper approximation is the union of equivalence classes that have a nonempty intersection with A. One may interpret $\underline{apr}, \overline{apr} : 2^U \longrightarrow 2^U$ as unary set-theoretic operators [19]. They are dual operators in the sense that $\underline{apr}(A) = \sim \overline{apr}(\sim A)$ and $\overline{apr}(A) = \sim \underline{apr}(\sim A)$. The system $(2^U, \sim, \underline{apr}, \overline{apr}, \cap, \cup)$ is called a Pawlak rough set algebra defined by the equivalence relation E. It may be viewed as an extension of the classical set algebra $(2^U, \sim, \cap, \cup)$.

Granules induced by an equivalence relation may be consider to be a special kind of neighborhoods. In general, one may use 1-neighborhood systems, in which each element has exactly one neighborhood. Such a neighborhood system can be conveniently constructed from a binary relation [20]. By extending Equation (7), we have the following two pairs of approximation operators [20]:

$$\begin{aligned}
\underline{apr}'_n(A) &= \bigcup\{n(x) \mid x \in U, n(x) \subseteq A\} \\
&= \{x \in U \mid \exists y[x \in n(y), n(y) \subseteq A]\}, \\
\overline{apr}'_n(A) &= \sim \underline{apr}'_n(\sim A) \\
&= \{x \in U \mid \forall y[x \in n(y) \Longrightarrow n(y) \cap A \neq \emptyset]\}, \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
\underline{apr}''_n(A) &= \sim \overline{apr}''_n(\sim A) \\
&= \{x \in U \mid \forall y[x \in n(y) \Longrightarrow n(y) \subseteq A]\}, \\
\overline{apr}''_n(A) &= \bigcup\{n(x) \mid x \in U, n(x) \cap A \neq \emptyset\} \\
&= \{x \in U \mid \exists y[x \in n(y), n(y) \cap A \neq \emptyset]\}. \qquad (9)
\end{aligned}$$

In the above definitions, we have replaced $[x]_E$ by $n(x)$ in either the lower or the upper approximation operator. The other one is defined by duality so that the two operators are dual to each other. By the first definition, the lower approximation is the union of granules which are subsets of A, while the upper approximation is the complement of the lower approximation of the complement of A. By the second definition, the upper approximation is the union of granules which have a nonempty intersection with A, while the lower approximation is the complement of the upper approximation of the complement of A. Each definition seems to be reasonable. If we only consider reflexive 1-neighborhood system, we have [20]:

$$\underline{apr}''_n(A) \subseteq \underline{apr}'_n(A) \subseteq A \subseteq \overline{apr}'_n(A) \subseteq \overline{apr}''_n(A). \qquad (10)$$

One pair of approximations is tighter than the other pair of approximations. The set A lies between its lower and upper approximations. Additional properties of the approximation operators can be found in a recent paper the author [20].

In the above development of rough sets, we start from a precise concept defined by a subset of the universe U. Relative to granules, a precise concept becomes vague and must be approximated. This formulation clearly identifies one possible sources of uncertainty. Such a model may have many practical applications. In many tasks, such as classification, diagnosis, reasoning, and learning, we normally focus on subsets (i.e., granules) of the universe sharing some special features, rather than on individuals. The granules may be understood as being the basic concepts or elementary knowledge. All other concepts and knowledge must be represented using such granulated information and knowledge. Two different, but related, approximations have been proposed, each is intuitively appealing.

3.3 Qualitative fuzzy sets

A fuzzy set $\mathcal{A}$ of U is defined by a membership function: $\mu_{\mathcal{A}} : U \longrightarrow [0,1]$. There are many definitions for fuzzy set complement, intersection, and union. With the min-max system proposed by Zadeh [23], fuzzy set operations are defined component-wise as:

$$\begin{aligned} \mu_{\sim\mathcal{A}}(x) &= 1 - \mu_{\mathcal{A}}(x), \\ \mu_{\mathcal{A}\cap\mathcal{B}}(x) &= \min[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)], \\ \mu_{\mathcal{A}\cup\mathcal{B}}(x) &= \max[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)]. \end{aligned} \tag{11}$$

In general, fuzzy set intersection and union may be defined in terms of t-norms and t-conorms [6]. By choosing different pairs of t-norms and t-conorms, one can derive distinct fuzzy set systems.

A crisp set can be regarded as a degenerated fuzzy set in which the membership function is restricted to the extreme points $\{0,1\}$ of $[0,1]$. In this case, the membership function is also referred to as a characteristic function. A fuzzy set can be related to a family of crisp sets through the notions of α-level sets. Given a number $\alpha \in [0,1]$, an α-cut, or α-level set, of a fuzzy set is defined by:

$$\mathcal{A}_{\alpha} = \{x \in U \mid \mu_{\mathcal{A}}(x) \geq \alpha\}, \tag{12}$$

which is a subset of U. A strong α-cut is defined by:

$$\mathcal{A}_{\alpha^+} = \{x \in U \mid \mu_{\mathcal{A}}(x) > \alpha\}. \tag{13}$$

The 1-cut of a fuzzy set is called the core of the fuzzy set and the strong 0-cut is called the support. They are defined by:

$$\begin{aligned} core(\mathcal{A}) &= \{x \mid \mu_{\mathcal{A}}(x) \geq 1\}, \\ support(\mathcal{A}) &= \{x \mid \mu_{\mathcal{A}}(x) > 0\}. \end{aligned} \tag{14}$$

An element in the core may be interpreted as a typical instance of the fuzzy concept being modeled, an element not in the support is definitely not an instance, and an element in the support is somewhere in between. A fuzzy set is normal if the core is not empty. Using either α-cuts or strong α-cuts, a fuzzy set determines a family of nested subsets of U. Conversely, a fuzzy set $\mathcal{A}$ can be reconstructed from its α-level sets as follows:

$$\mu_{\mathcal{A}}(x) = \sup\{\alpha \mid x \in \mathcal{A}_{\alpha}\}. \tag{15}$$

This observation is commonly summarized by a representation theorem of fuzzy sets, which states that there is an one-to-one relationship between a fuzzy set and a family of crisp sets satisfying certain conditions [6, 12, 13]. An implication of the min-max system is that fuzzy set operations can be defined by set operations on α-level sets. They can be expressed by:

$$\begin{aligned} (\sim \mathcal{A})_\alpha &= \sim \mathcal{A}_{(1-\alpha)^+}, \\ (\mathcal{A} \cap \mathcal{B})_\alpha &= \mathcal{A}_\alpha \cap \mathcal{B}_\alpha, \\ (\mathcal{A} \cup \mathcal{B})_\alpha &= \mathcal{A}_\alpha \cup \mathcal{B}_\alpha. \end{aligned} \tag{16}$$

The α-level sets of fuzzy sets for intersection and union are obtained from the same α-level sets of the fuzzy sets involved. When an arbitrary pair of t-norm and t-conorm is used, it may be difficult to define such operations using set operations on α-level sets of fuzzy sets.

Unlike the rough set theory, the fuzzy set theory starts from vague concepts which cannot be represented by a single subset of the universe. The sources of such uncertainty are not considered in the model. Given a fuzzy set, one can easily find the set of its distinct α-cuts:

$$S(\mathcal{A}) = \{A_\alpha \mid \alpha \in [0, 1]\}. \tag{17}$$

Each α-cut may be considered as a granule representing the fuzzy concept under certain view. In this study, we are only interested the family of distinct α-cuts, instead of the actual membership values. Such a view of fuzzy sets is qualitative. This serves as a basis of the development of a theory of qualitative fuzzy sets. In order to distinguish them from the conventional fuzzy sets, we referred to the conventional fuzzy sets as quantitative fuzzy sets.

A qualitative fuzzy set is a subsystem of the power set of the universe. That is, a qualitative fuzzy set $\mathcal{A}$ is a subset of 2^U, namely, $\mathcal{A} \subseteq 2^U$. Intuitively, one may understand a qualitative fuzzy set as a neighborhood system of a typical instance (possibly imaginary) of the fuzzy concept being modeled. The core and support of a qualitative fuzzy set are defined by:

$$\begin{aligned} core(\mathcal{A}) &= \bigcap_{X \in \mathcal{A}} X, \\ support(\mathcal{A}) &= \bigcup_{X \in \mathcal{A}} X. \end{aligned} \tag{18}$$

A qualitative fuzzy set is normal if its core is not empty. By interpreting a qualitative fuzzy set as a neighborhood system of a typical instance (possibly imaginary) of the fuzzy concept, we can use neighborhood system operations to define fuzzy set operations. More specifically, for two fuzzy sets $\mathcal{A}$ and $\mathcal{B}$, from

Equation (3) we have:

$$\begin{aligned}\neg\mathcal{A} &= \{\sim X \mid X \in \mathcal{A}\},\\ \mathcal{A} \sqcap \mathcal{B} &= \{X \cap Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\},\\ \mathcal{A} \sqcup \mathcal{B} &= \{X \cup Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}.\end{aligned} \quad (19)$$

Properties of these operations are given in Section 2.2. With respect to the ordering relation introduced in Section 2.3, one may also define qualitative membership. Given a qualitative fuzzy set $\mathcal{A}$, we may define an ordering relation $\prec$ representing a qualitative belongingness [1]. For two elements $a, b \in U$ with $a \prec b$, we may say that a belongs to $\mathcal{A}$ "more than" b belongs to $\mathcal{A}$. In other words, a is closer to a typical instance of the fuzzy concept modeled by $\mathcal{A}$ than b.

The use of a family of crisp sets to represent a fuzzy concept has been proposed and studied by many authors. Klir [5] proposed a formulation of fuzzy sets based on modal logic. In this model, a vague concept is characterized by some, possibly different, crisp sets. Let $W = \{w_1, \ldots, w_n\}$ denote a set of n possible worlds or states. With respect to W, a vague concept is represented by n crisp sets:

$$\mathcal{A} = (A_{w_1}, \ldots, A_{w_n}). \quad (20)$$

In each world w_i, the vague concept is described precisely by a crisp set A_{w_i}. The vagueness is captured by distinct representations of the same concept in different worlds. The set of all families of n crisp sets is given by the n-fold Cartesian product of 2^U, namely, $\prod_n 2^U = 2^U \times \ldots \times 2^U$ (n repetitions). We define set-theoretic operations on $\prod_n 2^U$ component-wise as follows: for $\mathcal{A} = (A_{w_1}, \ldots, A_{w_n})$ and $\mathcal{B} = (B_{w_1}, \ldots, B_{w_n})$,

$$\begin{aligned}&\sim \mathcal{A} = (\sim A_{w_1}, \ldots, \sim A_{w_n}),\\ &\mathcal{A} \cap \mathcal{B} = (A_{w_1} \cap B_{w_1}, \ldots, A_{w_n} \cap B_{w_n}),\\ &\mathcal{A} \cup \mathcal{B} = (A_{w_1} \cup B_{w_1}, \ldots, A_{w_n} \cup B_{w_n}).\end{aligned} \quad (21)$$

The system $(\prod_n 2^U, \sim, \cap, \cup)$ may be interpreted as the n-fold product of classical set algebra $(2^U, \sim, \cap, \cup)$. A similar approach was used by Kruse, Schwecke, and Heinsohn [8], in which each possible world is referred to as a context in a framework consisting of layered contexts. Kononov [7] studied a similar system. Negoiţă and Ralescu [12] investigated a more general system based on the notion of L-flou subsets. Like the representation of fuzzy sets using α-cuts, in those studies vague concepts are represented by the same number of subsets (possibly the same) of universe. Operations are defined component-wise using standard set-theoretic operations. However, in our formulation of qualitative fuzzy sets, we consider distinct subsets of the universe. No constraints are imposed on the number of subsets in representing a qualitative fuzzy set. Although operations on qualitative fuzzy sets are defined based on standard set-theoretic operations, they are not defined component-wise.

For a given quantitative fuzzy set, one can obtain a qualitative fuzzy set through its α-cuts. Two different quantitative fuzzy sets may produce the same qualitative fuzzy set. In addition, the qualitative fuzzy set obtained from a quantitative fuzzy set consists of a family of nested subsets of U. In general, an arbitrary qualitative fuzzy set may not be obtainable from a quantitative fuzzy set. The relationships between quantitative and qualitative fuzzy sets may be extended to L-fuzzy sets, i.e., lattice-based fuzzy sets [12]. It will also be useful to study the structures of a qualitative fuzzy set system.

4 Conclusion

A reflexive neighborhood of an element in a universe is a granule consisting of elements drawn towards that element by similarity. A neighborhood system of the element is a nonempty family of such granules. By lifting set-theoretic operations to families of subsets, one may define set-theoretic operations on neighborhood systems. This established a basis of a set-theoretic framework for granular computing. Both rough sets and fuzzy sets may be understood within the proposed framework.

In the theory of rough sets, granules are the basic concepts or elementary knowledge. All other concepts and knowledge must be represented using such granulated information and knowledge. A precise concept is defined by a subset of the universe U. When a precise concept is viewed relative to granules, it becomes vague and must be approximated. This formulation clearly identifies one possible sources of uncertainty. Such a model may be applied to situations where generalization plays an important role. A generalization is normally made based on subsets (i.e., granules) of the universe sharing some special features. Two related but different rough set approximations may be used.

From a quantitative fuzzy set, one may obtain a qualitative fuzzy set through the α-cuts of the former. In general, a qualitative fuzzy set is defined as a family of nonempty subsets of the universe. It may be understood as a neighborhood system of a typical instance (possibly imaginary) of the fuzzy concept being modeled. Operations on qualitative fuzzy sets are defined based on operations on neighborhood systems. The ordering induced by a qualitative fuzzy set represents qualitative belongingness. In general, an arbitrary qualitative fuzzy set may not be obtainable from a quantitative fuzzy set. As future work, it may be interesting to examine the relationships between quantitative and qualitative L-fuzzy sets. It may also be useful to study the structures of a qualitative fuzzy set system.

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