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On Generalizing Pawlak Approximation Operators

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Abstract. This paper reviews and discusses generalizations of Pawlak rough set approximation operators in mathematical systems, such as topological spaces, closure systems, lattices, and posets. The structures of generalized approximation spaces and the properties of approximation operators are analyzed.

1 Introduction

In the development of the theory of rough sets, approximation operators are typically defined by using equivalence relations [10, 11]. Researchers have proposed many generalized notions of approximation operators [2, 12, 15, 17, 19, 20]. Based on the results of these studies, we review and discuss generalizations of Pawlak rough set approximation operators, and show their connections with other mathematical systems.

We interpret the rough set theory as an extension of set theory with two additional unary set-theoretic operators referred to as approximation operators [16, 18]. Such an interpretation is consistent with interpreting modal logic as an extension of classical two-valued logic with two added unary operators [5]. With respect to an equivalence relation on a finite and nonempty universe, one can construct a subsystem of the power set of the universe, namely, the σ-algebra or the topology generated by the equivalence classes. Every subset of the universe is approximated by two sets of the subsystem. By generalizing this formulation, one may obtain generalized approximation operators. The resulting systems are related to topological spaces and closure systems. These systems are well-known in logic and algebraic literature [3, 7, 9, 13, 14]. We will first review and apply the relevant results for the present study of approximation operators. Further generalizations of approximation operators are studied using posets based on the recent work of Cattaneo [2].

2 Set-theoretic Approximation Operators

In this section, we apply results from the algebraic study of logic [3, 7, 13, 9, 14] to Pawlak type approximation operators. A common formulation is adopted from

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a recent paper by Cattaneo [2]. Two subsystems of the power set of a universe are considered. They are the family of inner definable subsets of the universe and the family of outer definable subsets. An arbitrary subset of the universe is approximated by an inner definable subset and an outer definable subset.

Let $E \subseteq U \times U$ be an equivalence relation on a finite and nonempty universe U. That is, the relation E is reflexive, symmetric, and transitive. The pair $apr = (U, E)$ is called a Pawlak approximation space. The equivalence relation E partitions the universe U into disjoint subsets called equivalence classes. Elements in the same equivalence class are said to be indistinguishable. Equivalence classes of E are called elementary sets. A union of elementary sets is called a definable (composed) set [10, 11]. The empty set is considered to be a definable set [16]. The family of all definable sets is denoted by $\mathrm{Def}(U)$. It is an σ-algebras of subsets of U. A Pawlak approximation space defines uniquely a topological space $(U, \mathrm{Def}(U))$, in which $\mathrm{Def}(U) \subseteq 2^U$ is the family of all open and closed sets [10].

For a subset $X \subseteq U$, one can approximate X by a pair of subsets of U. The lower approximation $i(X)$ is the greatest definable set contained in X, and the upper approximation $c(X)$ is the least definable set containing X. They correspond to the interior and closure of X in the topological space $(U, \mathrm{Def}(U))$. Thus we have the definition:

$$
\begin{aligned}
(\mathbf{P}) \qquad i(X) &= \bigcup \{Y \mid Y \in \mathrm{Def}(U), Y \subseteq X\}, \\
c(X) &= \bigcap \{Y \mid Y \in \mathrm{Def}(U), X \subseteq Y\}.
\end{aligned}
$$

One may interpret $i, c : 2^U \longrightarrow 2^U$ as unary set-theoretic operators [8, 15]. They are dual operators in the sense that $i(X) = \neg c(\neg X)$ and $c(X) = \neg i(\neg X)$. The system $(2^U, \neg, i, c, \cap, \cup)$ is called a Pawlak rough set algebra. It is an extension of the classical set algebra $(2^U, \neg, \cap, \cup)$.

Pawlak approximation operators have the following properties:

$$
\begin{aligned}
&(\mathrm{i1}) \quad i(X \cap Y) = i(X) \cap i(Y), \\
&(\mathrm{i2}) \quad i(X) \subseteq X, \\
&(\mathrm{i3}) \quad i(i(X)) = i(X), \\
&(\mathrm{i4}) \quad i(U) = U, \\
&(\mathrm{i5}) \quad c(X) = i(c(X)),
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{c1}) \quad c(X \cup Y) = c(X) \cup c(Y), \\
&(\mathrm{c2}) \quad X \subseteq c(X), \\
&(\mathrm{c3}) \quad c(c(X)) = c(X), \\
&(\mathrm{c4}) \quad c(\emptyset) = \emptyset, \\
&(\mathrm{c5}) \quad i(X) = c(i(X)).
\end{aligned}
$$

The above sets of properties are not independent. In fact, (i2) and (i5) imply (i3), and (c2) and (c5) imply (c3). The first four properties are the Kuratowski axioms for topological interior and closure operators. Axioms (i5) and (c5) show that an inner definable subset is also outer definable, and vice versa. Conversely, given a pair of approximation operators $i, c : 2^U \longrightarrow 2^U$ satisfying axioms (i1)-(i5) and axioms (c1)-(c5), respectively, their fixed elements:

$$\mathrm{Def}(U) = \{X \mid i(X) = X\} = \{X \mid c(X) = X\}, \tag{1}$$

are the open, and the closed, sets of an 0-dimensional topological space.

In the formulation of Pawlak approximation operators, a special type of topological space is used, in which the set of inner definable subsets (open sets) is the same as the set of outer definable subsets (closed sets). For an arbitrary topological space, the family of open sets is different from the family of closed sets. This immediately leads to a generalization of Pawlak approximation operators. Let $(U, \mathrm{O}(U))$ be a topological space, where $\mathrm{O}(U) \subseteq 2^U$ is a family of subsets of U called open sets. The family of open sets contains $\emptyset$ and U, and is closed under union and finite intersection. The family of all closed sets $\mathrm{C}(U) = \{\neg X \mid X \in \mathrm{O}(U)\}$ contains $\emptyset$ and U, and is closed under intersection and finite union. Following Cattaneo [2], a pair of approximation operators is defined by:

$$(\mathbf{T}) \qquad \begin{aligned} i(X) &= \bigcup\{Y \mid Y \in \mathrm{O}(U), Y \subseteq X\}, \\ c(X) &= \bigcap\{Y \mid Y \in \mathrm{C}(U), X \subseteq Y\}. \end{aligned}$$

They satisfy axioms (i1)-(i4), and axioms (c1)-(c4), respectively. Conversely, given approximation operators, $i, c : 2^U \longrightarrow 2^U$, satisfying axioms (i1)-(i4) and axioms (c1)-(c4), the sets of their fixed points:

$$\begin{aligned} \mathrm{O}(U) &= \{X \mid i(X) = X\}, \\ \mathrm{C}(U) &= \{X \mid c(X) = X\}, \end{aligned} \tag{2}$$

are families of open and, respectively closed, sets of a topological space.

The notion of closed sets in a topological space may be further generalized. A family $\mathcal{C}(U)$ of subsets of U is called a closure system if it contains U and is closed under intersection [3]. By collecting the complements of members of $\mathcal{C}(U)$, we obtain another system $\mathcal{O}(U) = \{\neg X \mid X \in \mathcal{C}(U)\}$. According to properties of $\mathcal{C}(U)$, the system $\mathcal{O}(U)$ contains the empty set $\emptyset$ and is closed under union. In this case, we define two approximation operators in a closure system:

$$(\mathbf{C}) \qquad \begin{aligned} i(X) &= \bigcup\{Y \mid Y \in \mathcal{O}(U), Y \subseteq X\}, \\ c(X) &= \bigcap\{Y \mid Y \in \mathcal{C}(U), X \subseteq Y\}. \end{aligned}$$

They satisfy axioms (i2) and (i3), and axioms (c2) and (c3), as well as the following weaker version of (i1) and (c1):

$$\begin{aligned} &\text{(i0)} \quad \text{If } X \subseteq Y, \text{ then } i(X) \subseteq i(Y), \\ &\text{(c0)} \quad \text{If } X \subseteq Y, \text{ then } c(X) \subseteq c(Y). \end{aligned}$$

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Conversely, for a closure operator $c : 2^U \longrightarrow 2^U$ satisfying axioms (c0), (c2), and (c3), the set of its fixed points:

$$\mathcal{C}(U) = \{X \mid c(X) = X\}, \tag{3}$$

is a closure system. Similar results can be stated between the system $\mathcal{O}(U)$:

$$\mathcal{O}(U) = \{X \mid i(X) = X\}, \tag{4}$$

and the dual operator $i(X) = \neg(c(\neg(X))$.

In defining set-theoretic approximation operators, three definitions (**P**), (**T**), and (**C**), in the order of generality, are used. For this formulation, inner definable sets must be closed under union and outer definable sets must be closed under intersection. A closure system is therefore the most generalized structure, and one cannot generalize set-theoretic approximation operators further under the same formulation.

3 Approximation Operators in Lattices

The power set of the universe is a special lattice. The results of the last section can be generalized as follows.

Suppose $(\mathcal{B}, \neg, \wedge, \vee, 0, 1)$ is a finite Boolean algebra and $(\mathcal{B}_0, \neg, \wedge, \vee, 0, 1)$ is a sub-Boolean algebra. By using (**P**), one may approximate an element of $\mathcal{B}$ using elements of $\mathcal{B}_0$:

$$(\mathbf{LP}) \qquad \begin{aligned} i(x) &= \bigvee \{y \mid y \in \mathcal{B}_0, y \leq x\}, \\ c(x) &= \bigwedge \{y \mid y \in \mathcal{B}_0, x \leq y\}. \end{aligned}$$

Any finite Boolean algebra is a complete Boolean algebra, and hence the above definition is well defined. Operators i and c satisfy the axioms:

$$\begin{aligned}
&(\text{i1}) \qquad i(x \wedge y) = i(x) \wedge i(y), \\
&(\text{i2}) \qquad i(x) \leq x, \\
&(\text{i3}) \qquad i(i(x)) = i(x), \\
&(\text{i4}) \qquad i(1) = 1, \\
&(\text{i5}) \qquad c(x) = i(c(x)),
\end{aligned}$$

and

$$\begin{aligned}
&(\text{c1}) \qquad c(x \vee y) = c(x) \vee c(y), \\
&(\text{c2}) \qquad x \leq c(x), \\
&(\text{c3}) \qquad c(c(x)) = c(x), \\
&(\text{c4}) \qquad c(0) = 0, \\
&(\text{c5}) \qquad i(x) = c(i(x)).
\end{aligned}$$

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Conversely, one may define a pair of approximation operators directly, and use their fixed points as inner and outer definable elements. Gehrke and Walker [4] considered a more generalized definition in which the Boolean algebra $\mathcal{B}$ is replaced by a completely distributive lattice. Like the Pawlak rough set model, one subsystem is used.

Consider a subsystem $\mathrm{O}(\mathcal{B})$ of $\mathcal{B}$ satisfying the following axioms:

$$
\begin{array}{ll}
\text{(O1)} & 0 \in \mathrm{O}(\mathcal{B}), 1 \in \mathrm{O}(\mathcal{B}); \\
\text{(O2)} & \text{For any subsystem } \mathcal{D} \subseteq \mathrm{O}(\mathcal{B}), \text{if there exists a least upper bound} \\
 & LUB(\mathcal{D}) = \bigvee \mathcal{D}, \text{ then it belongs to } \mathrm{O}(\mathcal{B}); \\
\text{(O3)} & \mathrm{O}(\mathcal{B}) \text{ is closed under finite meet, i.e.,} \\
 & \text{for any } x, y \in \mathrm{O}(\mathcal{B}), \text{ we have } x \wedge y \in \mathrm{O}(\mathcal{B}).
\end{array}
$$

For a finite Boolean algebra, axiom (O2) in fact states that the system $\mathrm{O}(\mathcal{B})$ is closed under join. Elements of $\mathrm{O}(\mathcal{B})$ are referred to as inner definable elements. The complement of an inner definable element is called an outer definable element. The set of outer definable elements $\mathrm{C}(\mathcal{B}) = \{\neg x \mid x \in \mathrm{O}(\mathcal{B})\}$ is characterized by the axioms:

$$
\begin{array}{ll}
\text{(C1)} & 0 \in \mathrm{C}(\mathcal{B}), 1 \in \mathrm{C}(\mathcal{B}); \\
\text{(C2)} & \text{For any subsystem } \mathcal{D} \subseteq \mathrm{C}(\mathcal{B}), \text{if there exists a greatest lower bound} \\
 & GLB(\mathcal{D}) = \bigwedge \mathcal{D}, \text{ then it belongs to } \mathrm{C}(\mathcal{B}); \\
\text{(C3)} & \mathrm{C}(\mathcal{B}) \text{ is closed under finite join, i.e.,} \\
 & \text{for any } x, y \in \mathrm{C}(\mathcal{B}), \text{ we have } x \vee y \in \mathrm{C}(\mathcal{B}).
\end{array}
$$

From the sets of inner and outer definable elements, we define the following approximation operators:

$$
\begin{array}{ll}
(\mathbf{LT}) & i(x) = \bigvee \{y \mid y \in \mathrm{O}(\mathcal{B}), y \leq x\}, \\
 & c(x) = \bigwedge \{y \mid y \in \mathrm{C}(\mathcal{B}), x \leq y\}.
\end{array}
$$

They satisfy axioms (i1)-(i4), and axioms (c1)-(c4), respectively, and are the topological interior and closure operators. The sets of fixed points of i and c are inner and outer definable elements, respectively. The system $(\mathcal{B}, \neg, i, c, \wedge, \vee, 0, 1)$ is a topological Boolean algebra [14], which is an extension of Boolean algebra with added operators.

Let $(L, \leq, 0, 1)$ be a bounded lattice. Suppose $\mathrm{O}(L)$ is a subset of L such that it contains 0 and is closed under join, and $\mathrm{C}(L)$ a subset of L such that it contains 1 and is closed under meet. They are complete lattices, although the meet of $\mathrm{O}(L)$ and the join of $\mathrm{C}(L)$ may be different from that of L. Based on these two systems, we can define two approximation operators as follows:

$$
\begin{array}{ll}
(\mathbf{LC}) & i(x) = \bigvee \{y \mid y \in \mathrm{O}(L), y \leq x\}, \\
 & c(x) = \bigwedge \{y \mid y \in \mathrm{C}(L), x \leq y\}.
\end{array}
$$

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The approximation operators satisfy axioms (i2) and (i3), axioms (c2) and (c3), as well as the following weaker version of (i1) and (c1), respectively:

$$\text{(i0)} \qquad \text{If } x \leq y, \text{ then } i(x) \leq i(y),$$
$$\text{(c0)} \qquad \text{If } x \leq y, \text{ then } c(x) \leq c(y).$$

The sets $\mathrm{O}(L)$ and $\mathrm{C}(L)$ are the fixed points of i and c, respectively. The operator c is a closure operator [1]. $C(L)$ corresponds to the closure system in the set-theoretic framework. But since a lattice may not be complemented, we must explicitly consider both $\mathrm{O}(L)$ and $\mathrm{C}(L)$. That is, the system $(L, \mathrm{O}(L), \mathrm{C}(L))$, or equivalently the system (L, i, c), is used for the generalization of Pawlak approximation operators.

4 Approximation Operators in Posets

Instead of using a Boolean algebra or a lattice, one may use a poset. Such generalizations of Pawlak rough set model were considered by Iwinski [6], and were systematically studied by Cattaneo recently [2]. Let $(\Sigma, \leq, 0, 1)$ be a poset with respect to a partial order relation $\leq$ bounded by the least element 0 and the greatest element 1. If the sets of inner, and respectively outer, definable elements of Σ are chosen to be complete lattices, one may immediately use (**LC**) to define approximation operators. A similar idea is in fact used by Iwinski [6], although the definition is different from (**LC**). The formulation suggested by Cattaneo [2] is consistent with (**LC**).

For an arbitrary subset $X \subseteq \Sigma$, a least upper bound or a greatest lower bound of X may not exist. If a least upper bound of X exists, it is unique and is denoted by $LUB(X)$. Similarly, if a greatest lower bound of X exists, it is unique and is denoted by $GLB(X)$. Any subset Σ_0 of a poset Σ is itself a poset under the same order relation, and is called a subposet. In the subposet Σ_0, if the least upper bound of a subset $X \subseteq \Sigma_0$ exists, we denote it by $LUB_{\Sigma_0}(X)$ or simply $LUB(X)$ when Σ_0 is clear from the context. If the greatest lower bound of X exists, we denote it by $GLB_{\Sigma_0}(X)$ or simply $GLB(X)$. Both Boolean algebras and lattice can be understood as posets with additional properties. For the three definitions in the last section, approximation operators are defined through the LUB of a subsystem of inner definable elements, and the GLB of a subsystem of outer definable elements. Following the same argument, we will use two subposets $\mathrm{O}(\Sigma)$ and $\mathrm{C}(\Sigma)$ to represent inner and outer definable elements. But we must require that in some sense the subposet $\mathrm{O}(\Sigma)$ be closed with respect to LUB, and $\mathrm{C}(\Sigma)$ be closed with respect to GLB.

Given $x \in \Sigma$ and $Y \subseteq \Sigma$, the order ideal relative to Y generated by x is defined by:

$$\downarrow x|Y = \{y \in Y \mid y \leq x\}. \tag{5}$$

Dually, the order filter relative to Y generated by x is defined by:

$$\uparrow x|Y = \{y \in Y \mid x \leq y\}. \tag{6}$$

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With these notions, we can construct two families of subsets of $\mathrm{O}(\Sigma)$ and $\mathrm{C}(\Sigma)$:

$$\begin{aligned} \mathrm{SO}(\Sigma) &= \{\downarrow x|\mathrm{O}(\Sigma) \mid x \in \Sigma\}, \\ \mathrm{SC}(\Sigma) &= \{\uparrow x|\mathrm{C}(\Sigma) \mid x \in \Sigma\}. \end{aligned} \tag{7}$$

Each element of $\mathrm{SO}(\Sigma)$ is a subsystem of $\mathrm{O}(\Sigma)$, and each element of $\mathrm{SC}(\Sigma)$ is a subsystem of $\mathrm{C}(\Sigma)$.

For the definition of abstract approximation operators in posets, we adopt and generalize the proposal of Cattaneo [2]. However, the formulation is slightly different. In particular, we explicitly specify the structure of the set of inner definable and the structure of the set of outer definable elements. Consider a triple $(\Sigma, \mathrm{O}(\Sigma), \mathrm{C}(\Sigma))$ called an abstract approximation space, where Σ is a bounded poset, $\mathrm{O}(\Sigma)$ is assumed to be the set of inner definable elements, and $\mathrm{C}(\Sigma)$ is assumed to be the set of outer definable elements. The set of inner definable elements is characterized by the axioms:

(O1) $0 \in \mathrm{O}(\Sigma), 1 \in \mathrm{O}(\Sigma)$;

(O2*) With respect to $\mathrm{O}(\Sigma)$, the least upper bound of $\downarrow x|\mathrm{O}(\Sigma)$ exists and satisfies the condition :

$$LUB_{\mathrm{O}(\Sigma)}(\downarrow x|\mathrm{O}(\Sigma)) \leq x, \quad \text{for every } x \in \Sigma.$$

Axiom (O2*) suggests that $LUB_{\mathrm{O}(\Sigma)}(\downarrow x|\mathrm{O}(\Sigma))$ is also an inner definable element. That is, $\mathrm{O}(\Sigma)$ is closed under LUB at least for any subsystem of $\mathrm{SO}(\Sigma)$. For an arbitrary subset of $\mathrm{O}(\Sigma)$, the least upper bound may not exist. Hence the system $\mathrm{O}(\Sigma)$ may not be a lattice. The set of outer definable elements is defined by the axioms:

(C1) $0 \in \mathrm{C}(\Sigma), 1 \in \mathrm{C}(\Sigma)$;

(C2*) With respect to $\mathrm{C}(\Sigma)$, the greatest lower bound of $\uparrow x|\mathrm{C}(\Sigma)$ exists and satisfies the condition :

$$x \leq GLB_{\mathrm{C}(\Sigma)}(\uparrow x|\mathrm{C}(\Sigma)), \quad \text{for every } x \in \Sigma.$$

The subposet $\mathrm{C}(\Sigma)$ is closed under GLB for at least any subsystem of $\mathrm{SC}(\Sigma)$. It may not be a lattice. The two systems $\mathrm{O}(\Sigma)$ and $\mathrm{C}(\Sigma)$ are usually different subsets of Σ. The set $\mathrm{O}(\Sigma) \cap \mathrm{C}(\Sigma)$ consists of those elements which are both inner and outer definable.

An element of Σ may be approximated by a pair of inner and outer definable elements from $\mathrm{O}(\Sigma)$ and $\mathrm{C}(\Sigma)$. We define a pair of inner and outer approximation operators, $i : \Sigma \longrightarrow \mathrm{O}(\Sigma)$ and $c : \Sigma \longrightarrow \mathrm{C}(\Sigma)$, as follows:

$$\begin{aligned} (\mathbf{PC}) \qquad i(x) &= LUB(\downarrow x|\mathrm{O}(\Sigma)) \\ &= LUB(\{y \in \mathrm{O}(\Sigma) \mid y \leq x\}), \\ c(x) &= GLB(\uparrow x|\mathrm{C}(\Sigma)) \\ &= GLB(\{y \in \mathrm{C}(\Sigma) \mid x \leq y\}), \end{aligned}$$

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where LUB and GLB are defined with respect to $\mathrm{O}(\Sigma)$ and $\mathrm{C}(\Sigma)$. By definition, $i(x)$ is the best approximation of x from below using the inner definable elements $\mathrm{O}(\Sigma)$, while $c(x)$ is the best approximation of x from above using outer definable elements $\mathrm{C}(\Sigma)$. More specifically, i satisfies axioms (i0) and (i2)-(i4), and c satisfies axioms (c0) and (c2)-(c4). Assume $x \leq y$. We have $\downarrow x|\mathrm{O}(\Sigma) \subseteq \downarrow y|\mathrm{O}(\Sigma)$. Hence, $i(x) = LUB(\downarrow x|\mathrm{O}(\Sigma)) \leq LUB(\downarrow y|\mathrm{O}(\Sigma)) = i(y)$, namely, (i0) holds. By (O2*), i satisfies (i2). For any $y \in \mathrm{O}(\Sigma)$, we have $y \in \downarrow x|\mathrm{O}(\Sigma)$. This implies $y \leq i(y)$. By combining it with (i2), we have $i(y) = y$ for any $y \in \mathrm{O}(\Sigma)$. For any $x \in \Sigma$, $i(x) \in \mathrm{O}(\Sigma)$. Thus, $i(i(x)) = i(x)$, namely (i3) holds. By the assumption $1 \in \mathrm{O}(\Sigma)$ and the definition of i, it follows that i satisfies (i4). Similarly, we can show that c obeys (c0) and (c2)-(c4).

Alternatively, we may define an abstract approximation space by a triple (Σ, i, c), where i and c are mappings characterized by axioms (i0) and (i2)-(i4), and axioms (c0) and (c2)-(c4), respectively. The sets of i-fixed and c-fixed elements:

$$\begin{aligned} \mathrm{O}(\Sigma) &= \{x \in \Sigma \mid i(x) = x\}, \\ \mathrm{C}(\Sigma) &= \{x \in \Sigma \mid c(x) = x\}, \end{aligned} \tag{8}$$

are the sets of inner and outer definable elements, respectively. By axioms (i2) and (i4), it can be easily verified that $0, 1 \in \mathrm{O}(\Sigma)$. Consider an element $x \in \Sigma$. Suppose $y \in \downarrow x|\mathrm{O}(\Sigma)$. We have $y \leq x$ and $y = i(y)$. By axiom (i0), it follows $y = i(y) \leq i(x)$. Thus, $i(x)$ is an upper bound of $\downarrow x|\mathrm{O}(\Sigma)$. Now, we want to show that it is in fact the least upper bound. Suppose z is an upper bound of $\downarrow x|\mathrm{O}(\Sigma)$. We have $y \leq z$ for all $y \in \downarrow x|\mathrm{O}(\Sigma)$. By axiom (i2), $i(x) \leq x$. By axiom (i3), $i(x) \in \mathrm{O}(\Sigma)$. Therefore, $i(x) \in \downarrow x|\mathrm{O}(\Sigma)$. It immediately follows that $i(x) \leq z$. This implies that (O2*) holds. Similarly, one can show that C is the family of outer definable elements satisfying axioms (C1) and (C2*).

In the previous formulation of approximation operators, two methods have been used. One starts from the system $(\Sigma, \mathrm{O}(\Sigma), \mathrm{C}(\Sigma))$ where the two subposets $\mathrm{O}(\Sigma)$ and $\mathrm{C}(\Sigma)$) are given specific structures. From this system one can define two approximation operators, i and c, enjoying certain properties. Dually, the second method starts from a pair of approximation operators, namely, the system (Σ, i, c) satisfying some axioms, and a system $(\Sigma, \mathrm{O}(\Sigma), \mathrm{C}(\Sigma))$ is recovered by the sets of fixed points of i and c. In the formulation of Cattaneo [2], the structures of two subposets are stated implicitly by using the approximation operators. Our formulation avoided such a problem. This makes our discussion of approximation operators to be conform to the commonly used approaches for the study of approximation operators in systems such as topological spaces and closure systems.

Approximation operators in posets can be further generalized. Suppose that the set of inner definable elements $\mathrm{O}(\Sigma)$ satisfies the axiom:

$$(\mathrm{O1}^*) \qquad 0 \in \mathrm{O}(\Sigma),$$

and axiom (O2*). The approximation operator i defined by (**PC**) only satisfies axioms (i0), (i2), and (i3). Similarly, if the set of outer definable elements $\mathrm{C}(\Sigma)$

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satisfies the axiom:

$$(\mathrm{C1}^*) \qquad 1 \in \mathrm{C}(\Sigma),$$

and axiom ($\mathrm{C2}^*$), the approximation operator c defined by (**PC**) only satisfies axioms (c0), (c2), and (c3). They correspond to approximation operators in closure systems. A subposet $\mathrm{C}(\Sigma)$ is said to be a closure system if it satisfies axioms ($\mathrm{C1}^*$) and ($\mathrm{C2}^*$). In the set-theoretic setting, a closure system must be closed under intersection for any of its subsystem. A closure system on a poset is closed under *GLB* for only certain subsystems.

Suppose the set of inner definable elements $\mathrm{O}(\Sigma)$ satisfies the axioms ($\mathrm{O1}^*$) and ($\mathrm{O2}^{**}$): for $x \in \Sigma$,

$$(\mathrm{O2}^{**}) \qquad \text{With respect to } \mathrm{O}(\Sigma), \text{ the least upper bound of } \downarrow x|\mathrm{O}(\Sigma) \text{ exists},$$

and the set of outer definable elements $\mathrm{C}(\Sigma)$ satisfies the axioms ($\mathrm{C1}^*$) and ($\mathrm{C2}^{**}$): for $x \in \Sigma$,

$$(\mathrm{C2}^{**}) \qquad \text{With respect to } \mathrm{C}(\Sigma), \text{ the greatest lower bound of } \uparrow x|\mathrm{C}(\Sigma) \text{ exists}.$$

In this case, approximation operator i defined by (**PC**) satisfies axioms (i0), (i3), and $i(0) = 0$. Approximation operator c satisfies axioms (c0), (c3), and $c(1) = 1$.

A more detailed and systematic study of approximation operators in special types of lattice and posets, as well as examples, can be found in a recent paper by Cattaneo [2]. A different formulation of approximation operators in poset can be found in an earlier paper by Iwinski [6].

5 Conclusion

In generalizing Pawlak approximation operators, we have considered four systems. From more particular instantiations to more general cases, they are Pawlak approximation spaces (0-dimensional topological spaces), topological Boolean algebras (topological spaces), closure systems, and abstract approximation spaces. For the definition of approximation operators, two subsystems, corresponding to the set of inner definable elements and the set of outer definable elements, are used. An arbitrary element is approximated from blow by using inner definable elements through *LUB*, and from the above by using outer definable elements through *GLB*. In Pawlak approximation spaces, the set of inner definable elements is the same as the set of outer definable elements. It contains 0 and 1, and is closed under both *LUB* and *GLB*. For topological Boolean algebras, the set of inner definable elements is closed under *LUB* and finite *GLB*, while the set of outer definable elements is closed under *GLB* and finite *LUB*. For closure system, the set of inner definable elements is only closed under *LUB*, and the set of outer definable elements is only closed under *GLB*. For abstract approximation spaces, the set of inner definable elements is partially closed under *LUB* (i.e., for only some subsystems), while the set of outer definable elements is partially closed under GLB. The structure of the family of inner definable elements determines the properties of inner approximation operator. Likewise, the structure of the family of outer definable elements determines the properties of outer approximation operator. Dually, one may start from a pair of approximation operators. The set of inner definable elements can be obtained by the fixed points of the inner approximation operator, while the set of outer definable elements can be obtained by the fixed points of the outer approximation operator.

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