Three-Way Decision: An Interpretation of Rules in Rough Set Theory

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Abstract. A new interpretation of rules in rough set theory is introduced. According to the positive, boundary, and negative regions of a set, one can make a three-way decision: accept, abstain and reject. The three regions enable us to derive three types of decision rules, namely, positive rules for acceptance, boundary rules for indecision or delayed decision, and negative rules for rejection. Within the decision-theoretic rough set model, the associated costs of rules are analyzed.

1 Introduction

An important application of rough set theory is to induce classification or decision rules that indicate the decision class of an object based on its values on some condition attributes [6, 8]. A survey of existing studies shows that several interpretations of rules have been commonly used in rough sets research. They have led to a wide range of applications on one hand, and to some confusion on the other.

A decision class is a subset of a universe of objects and is approximated by a pair of definable sets with respect to a logic language [15]. Based solely on the attribute values of objects, the lower approximation consists of those objects that *certainly* belong to the decision class, and the upper approximation consists of those objects that only *possibly* belong to the decision class [6]. Accordingly, Grzymala-Busse [3] suggested that two categories of rules can be induced: "certain rules" from the lower approximation and "possible rules" from the upper approximation.

The lower and upper approximations divide the universe of objects into three pair-wise disjoint regions: the lower approximation as the positive region, the difference between the upper and lower approximations as the boundary region, and the complement of the upper approximation as the negative region. For objects in both positive and negative regions, we can make *deterministic decisions* about their memberships in the given decision class. We can only make *nondeterministic decisions* for objects in the boundary region. Based on this observation, Wong and Ziarko [11] proposed two types of rules: "deterministic decision rules" for the positive region and "undeterministic rules" for the boundary region.

One may associate probabilistic measures, such as accuracy, confidence, and coverage, to rules [10]. The accuracy and confidence of a deterministic rule is 1, namely, totally certain, and that of a nondeterministic rule is between 0 and 1 exclusively, namely, uncertain. Thus, Pawlak [7] referred to them, respectively, as "certain decision rules" and "uncertain decision rules."

Although other classifications and interpretations of rules have been considered in rough set theory, they are basically variations of the above three. Within the classical Pawlak rough set theory, these interpretations make perfect sense. They truthfully reflect the *qualitative, statistical, or syntactical* nature of rules, *certain* versus *possible* [3], *deterministic* versus *nondeterministic* [11], and *certain* versus *uncertain* [7]. In his book and earlier works, Pawlak [6] focused mainly on the positive region and certain rules, as they characterize the objects on which we can make consistent and correct decisions.

When the rough set model is generalized into probabilistic rough set models [9, 16–18], these qualitative interpretations of rules are no longer appropriate. An object in the probabilistic positive region does *not certainly* belong to the decision class, but with a *high probability*. Like a probabilistic rule from the probabilistic boundary region, a rule from a probabilistic positive region may be *uncertain* and *nondeterministic*. Some authors simply used the same notions of the classical rough set model in the probabilistic models, without considering their differences. There remains a semantics difficulty and confusion in interpreting two categories of probabilistic rules from, respectively, the probabilistic positive and boundary regions.

In earlier papers [16, 19], we argued that a solution can be sought from the *semantics* of rules, rather than their *syntactical* characteristics of being *certain* or *uncertain*. Rules are interpreted and classified based on their associated actions and decisions. With respect to the positive and boundary regions, we introduced the notions of *positive rules* and *boundary rules*. A positive rule makes a decision of acceptance and a boundary rules makes a decision of further-investigation [13, 19]. This paper contributes further on the topic. For the negative region, we introduce the notion of *negative rules*. A negative rule makes a decision of rejection. This new interpretation relates the rough set theory to many studies on three-way decision-making [1, 2, 12], and hence leads to more applications.

2 Three-way Decisions from Rough Set Approximations

Based on the three regions, three types of rules are introduced and analyzed.

2.1 Rules in the classical rough set model

Consider a simple knowledge representation scheme in which a finite set of objects is described by using a finite set of attributes. Formally, it can be defined by an information table M expressed as the tuple [6]:

$$M = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\}), \tag{1}$$

where U is a finite nonempty set of objects, At is a finite nonempty set of attributes, V_a is a nonempty set of values for an attribute $a \in At$, and I_a :

$U \longrightarrow V_a$ is an information or a description function. It is assumed that the mapping I_a is single-valued. In this case, the value of an object $x \in U$ on an attribute $a \in At$ is denoted by $I_a(x)$.

Given a subset of attributes $A \subseteq At$, we define an indiscernibility relation $ind(A)$ on U as follows [6]:

$$x \; ind(A) \; y \Longleftrightarrow \forall a \in A[I_a(x) = I_a(y)]. \tag{2}$$

That is, two objects x and y are indiscernible with respect to A if and only if they have exactly the same value on every attribute in A. The indiscernibility relation $ind(A)$ is an equivalence relation. The equivalence class containing object x is denoted by $[x]_{ind(A)}$, or simply $[x]_A$ and $[x]$ if no confusion arises.

The pair $apr = (U, ind(A))$ is called an approximation apace. For an arbitrary subset $X \subseteq U$, we obtain, by using equivalence classes, a pair of lower and upper approximations as follows [6]:

$$\begin{aligned} \underline{apr}(X) &= \{x \mid x \in U, [x] \subseteq X\}; \\ \overline{apr}(X) &= \{x \mid x \in U, [x] \cap X \neq \emptyset\}. \end{aligned} \tag{3}$$

Based on the rough set approximations of X, one can divide the universe U into three disjoint regions, the positive region $\mathrm{POS}(X)$, the boundary region $\mathrm{BND}(X)$, and the negative region $\mathrm{NEG}(X)$:

$$\begin{aligned} \mathrm{POS}(X) &= \underline{apr}(X), \\ \mathrm{BND}(X) &= \overline{apr}(X) - \underline{apr}(X), \\ \mathrm{NEG}(X) &= U - \mathrm{POS}(X) \cup \mathrm{BND}(X) = U - \overline{apr}(X) = (\overline{apr}(X))^c. \end{aligned} \tag{4}$$

One can say with *certainty* that any element $x \in \mathrm{POS}(X)$ belongs to X, and that any element $x \in \mathrm{NEG}(X)$ does not belong to X. One cannot decide with certainty whether or not an element $x \in \mathrm{BND}(X)$ belongs to X.

We are now ready to discuss the problem of decision and classification rule induction. For a classification problem, we divide the set of attributes into a subset of condition attributes C and a subset of decision attributes D, namely, $At = C \cup D$ and $C \cap D = \emptyset$. For simplicity, let $class = \{d_1, d_2, \ldots, d_m\}$ denote the m disjoint decision classes defined by the set of decision attributes D, where $d_i \subseteq U$ and $d_i \cap d_j = \emptyset$ for $i \neq j$. Each decision class $d_i \in class$ divides U into two disjoint sets $d = d_i$ and $d^c = U - d_i = \bigcup_{i \neq j} d_j$. Thus, we reformulate an m-class classification problem into m 2-class classification problems. In what follows, we focus on decision rules that distinguish d and d^c based on a subset of condition attributes $A \subseteq C$.

In the approximation space $apr = (U, ind(A))$, we can construct the positive, boundary, and negative regions of a decision class d. Accordingly, we can derive positive, boundary, and negative decision rules:

$$\begin{aligned} &\mathrm{Des}([x]) \longrightarrow_P \mathrm{Des}(d), \quad \text{for } [x] \subseteq \mathrm{POS}(d); \\ &\mathrm{Des}([x]) \longrightarrow_B \mathrm{Des}(d), \quad \text{for } [x] \subseteq \mathrm{BND}(d); \\ &\mathrm{Des}([x]) \longrightarrow_N \mathrm{Des}(d), \quad \text{for } [x] \subseteq \mathrm{NEG}(d); \end{aligned}$$

where Des(·) denotes the logic formula defining a set [6, 9]. For example, the set $[x]$ can be defined by the logic expression $\bigwedge_{a \in A}(a = I_a(x))$.

In spite of their similarity in form, the three types of rules have very different semantics interpretations, and each of them leads to a different decision. A positive rule allows us to *accept* an object to be a member of d and a negative rule allows us to *reject* an object to be a member of d. On the other hand, a boundary rule does not offer such a definite decision, and due to uncertainty and inconsistency we are forced to, or happy to, make an indecision or a delayed decision, which warrants a further investigation [13]. This alternative interpretation, based on three-way decision, seems to be more suitable for viewing rules in rough set theory. Pawlak, Wong, and Ziarko [9] considered a similar interpretation based on three-valued decisions, consisting of "yes", "no", and "do not know." Herbert and Yao [4] interpreted the positive and negative rules as providing "immediate decisions" of "yes" and "no", respectively, and boundary rules as providing "delayed decisions" of "wait-and-see." Li, Zhang and Swan [5] considered the three-way decision as providing "three distinct regions of relevance."

For a decision rule $\mathrm{Des}([x]) \longrightarrow_{\Lambda} \mathrm{Des}(d)$, where $\Lambda \in \{P, B, N\}$, we can associate a probabilistic measure called the accuracy or confidence of the rule as follows:

$$conf(\mathrm{Des}([x]) \longrightarrow_{\Lambda} \mathrm{Des}(d)) = Pr(d \mid [x]) = \frac{|[x] \cap d|}{|[x]|}, \tag{5}$$

where $|\cdot|$ denotes the cardinality of a set, and $Pr(d \mid [x])$ is the conditional probability of an object in d given that the object is in $[x]$, estimated by using cardinalities of sets. According to the confidence values, positive, boundary, and negative rules are defined by the conditions $conf = 1$, $0 < conf < 1$, and $conf = 0$, respectively. That is, the decisions of acceptance and rejection are made by using the two extreme values, 0 and 1, of probability. This qualitative categorization may be too restrictive to be practically useful, and hence probabilistic rough set models have been introduced.

2.2 Rules in probabilistic rough set models

Probabilistic rough set models allow a tolerance of inaccuracy in the lower and upper approximations, or the probabilistic positive, boundary, and negative regions. Based on the well established Bayesian decision procedure, the decision-theoretic rough set model provides systematic methods for deriving the most suitable thresholds on probabilities for defining the three regions [14, 16–18].

In formulating the decision-theoretic rough set model, we have a set of 2 states and a set of 3 actions for each state. For a subset $X \subseteq U$, the set of states is given by $\Omega = \{X, X^c\}$, indicating that an element is in X and not in X, respectively. For simplicity, we use the same symbol to denote both a subset X and the corresponding state. The set of actions regarding the state X is given by $\mathcal{A} = \{P, B, N\}$, where P, B, and N represent the three actions in classifying an object, namely, deciding $x \in \mathrm{POS}(X)$, deciding $x \in \mathrm{BND}(X)$, and deciding $x \in \mathrm{NEG}(X)$, respectively. The loss function regarding the risk or cost of actions in different states is given by the 3×2 matrix:

	X (P)	X^c (N)
P	λ_{PP}	λ_{PN}
B	λ_{BP}	λ_{BN}
N	λ_{NP}	λ_{NN}

In the matrix, λ_{PP}, λ_{BP} and λ_{NP} denote the losses incurred for taking actions P, B and N, respectively, when an object belongs to X, and λ_{PN}, λ_{BN} and λ_{NN} denote the losses incurred for taking the same actions when the object does not belong to X.

The expected losses for taking different actions for objects in $[x]$ are:

$$\begin{aligned} R(P|[x]) &= \lambda_{PP} Pr(X \mid [x]) + \lambda_{PN} Pr(X^c \mid [x]), \\ R(B|[x]) &= \lambda_{BP} Pr(X \mid [x]) + \lambda_{BN} Pr(X^c \mid [x]), \\ R(N|[x]) &= \lambda_{NP} Pr(X \mid [x]) + \lambda_{NN} Pr(X^c \mid [x]). \end{aligned} \tag{6}$$

The Bayesian decision procedure suggests the minimum-risk decision rules:

(P) If $R(P|[x]) \leq R(N|[x])$ and $R(P|[x]) \leq R(B|[x])$, decide $x \in \mathrm{POS}(X)$;
(B) If $R(B|[x]) \leq R(P|[x])$ and $R(B|[x]) \leq R(N|[x])$, decide $x \in \mathrm{BND}(X)$;
(N) If $R(N|[x]) \leq R(P|[x])$ and $R(N|[x]) \leq R(B|[x])$, decide $x \in \mathrm{NEG}(X)$.

Tie-breaking criteria should be added so that each object is put into only one region. Since $Pr(X \mid [x]) + Pr(X^c \mid [x]) = 1$, we can simplify the rules based only on the probabilities $Pr(X \mid [x])$ and the loss function λ.

Consider a special kind of loss functions with $\lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$. That is, the loss of classifying an object x belonging to X into the positive region $\mathrm{POS}(X)$ is less than or equal to the loss of classifying x into the boundary region $\mathrm{BND}(X)$, and both of these losses are strictly less than the loss of classifying x into the negative region $\mathrm{NEG}(X)$. The reverse order of losses is used for classifying an object not in X. We further assume that a loss function satisfies the condition [16]:

$$(\lambda_{PN} - \lambda_{BN})(\lambda_{NP} - \lambda_{BP}) > (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}). \tag{7}$$

Under the two assumptions, we have the simplified rules [16]:

(P1) If $Pr(X \mid [x]) \geq \alpha$, decide $x \in \mathrm{POS}(X)$;
(B1) If $\beta < Pr(X \mid [x]) < \alpha$, decide $x \in \mathrm{BND}(X)$;
(N1) If $Pr(X \mid [x]) \leq \beta$, decide $x \in \mathrm{NEG}(X)$;

$$\begin{aligned} \alpha &= \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}, \\ \beta &= \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}, \end{aligned} \tag{8}$$

with $1 \geq \alpha > \beta \geq 0$.

The (α, β)-probabilistic positive, boundary and negative regions are given by rules (P1), (B1), and (N1), respectively, as:

$$\begin{aligned} \mathrm{POS}_{(\alpha,\beta)}(X) &= \{x \in U \mid Pr(X \mid [x]) \geq \alpha\}, \\ \mathrm{BND}_{(\alpha,\beta)}(X) &= \{x \in U \mid \beta < Pr(X \mid [x]) < \alpha\}, \\ \mathrm{NEG}_{(\alpha,\beta)}(X) &= \{x \in U \mid Pr(X \mid [x]) \leq \beta\}. \end{aligned} \tag{9}$$

The (α, β)-probabilistic lower and upper approximations are defined by:

$$\begin{aligned} \underline{apr}_{(\alpha,\beta)}(X) &= \{x \in U \mid Pr(X \mid [x]) \geq \alpha\}, \\ \overline{apr}_{(\alpha,\beta)}(X) &= \{x \in U \mid Pr(X \mid [x]) > \beta\}. \end{aligned} \tag{10}$$

A detailed derivations of existing probabilistic models are given in other papers [16, 17].

According the three probabilistic regions of a decision class d, we have the following positive, boundary and negative decision rules:

$$\begin{aligned} &\mathrm{Des}([x]) \longrightarrow_P \mathrm{Des}(d), \quad \text{for } [x] \subseteq \mathrm{POS}_{(\alpha,\beta)}(d); \\ &\mathrm{Des}([x]) \longrightarrow_B \mathrm{Des}(d), \quad \text{for } [x] \subseteq \mathrm{BND}_{(\alpha,\beta)}(d); \\ &\mathrm{Des}([x]) \longrightarrow_N \mathrm{Des}(d), \quad \text{for } [x] \subseteq \mathrm{NEG}_{(\alpha,\beta)}(d). \end{aligned}$$

Depending on the values of α and β, an equivalence class may produce more than one positive rule. For $\alpha \geq 0.5$, each equivalence class produces at most one positive rule. Similar results can be stated for other types of rules.

Several important remarks on probabilistic rules are in order. First, unlike rules in the classical rough set theory, all three types of rules may be *uncertain* and *nondeterministic*. These rules again lead to a three-way decision, based on two threshold values α and β. Intuitively, they represent the levels of our tolerance in making incorrect decisions. For positive rules, the error rate of accepting a non-member of d as a member of d is below $1 - \alpha$. Conversely, the error rate of rejecting a member of d as a non-member of d is below β. When the conditional probability is too low for acceptance but too high for rejection, we choose a boundary rule for an indecision or a delayed decision. In practical situations, this normally implies a further investigation. The introduction of the third choice perhaps illustrates better the philosophy and power of rough set theory. For classification problems with more than two classes, one may not be interested in negative rules. The objects in the negative region of one class may be in the positive and boundary regions of other classes. For this reason, we only considered positive and boundary rules in earlier studies [16, 19]. Retrospectively, the notion of three-way decision seems to be more accurate and appropriate for interpreting rules in rough set theory.

Second, the conditional probability $Pr(d \mid [x])$ is the accuracy and confidence of a rule. The three types of rules are indeed characterized by their accuracy and confidence. One can find the conditions on the loss function so that we can obtain the conditions $\alpha = 1$ and $\beta = 0$ of the classical rough set model. However, rules in classical model are of qualitative nature and rules in probabilistic models are quantitative. In addition, the semantics differences of the three types of rules can be easily explained by their associated different costs:

$$\begin{aligned} \text{positive rule}: &\quad \mathit{conf} * \lambda_{PP} + (1 - \mathit{conf}) * \lambda_{PN}, \\ \text{boundary rule}: &\quad \mathit{conf} * \lambda_{BP} + (1 - \mathit{conf}) * \lambda_{BN}, \\ \text{negative rule}: &\quad \mathit{conf} * \lambda_{NP} + (1 - \mathit{conf}) * \lambda_{NN}, \end{aligned} \tag{11}$$

where $\mathit{conf} = Pr(d \mid [x])$ for rule $\mathrm{Des}([x]) \longrightarrow_{\Lambda} \mathrm{Des}(d)$, $\Lambda \in \{P, B, N\}$. In the special case where we assume zero cost for a correct classification, namely, $\lambda_{PP} = \lambda_{NN} = 0$, costs associated with rules can be simplified to:

$$\begin{aligned} \text{positive rule}: &\quad (1 - \mathit{conf}) * \lambda_{PN}, \\ \text{boundary rule}: &\quad \mathit{conf} * \lambda_{BP} + (1 - \mathit{conf}) * \lambda_{BN}, \\ \text{negative rule}: &\quad \mathit{conf} * \lambda_{NP}. \end{aligned} \tag{12}$$

They are much easier to understand in terms of misclassification errors.

Third, in applications of probabilistic rough set models, one may directly supply the parameters α and β based on an intuitive understanding the levels of tolerance for errors. This means that one, in fact, uses an intermediate result of the decision-theoretic rough set model. Such ad hoc uses of parameters α and β may be largely due to an unawareness of the well-established Bayesian decision procedure. More often than not, one may find it much easier to give loss functions that can be related to more intuitive terms such as costs, benefits, and risks, than to give abstract threshold values. This is particular true in situations where the costs can be translated into monetary values.

Fourth, the new interpretation of rules makes it easy to relate rough set theory to many studies on statical inference and decision involving a three-way decision [1, 2, 12]. Woodward and Naylor[12] discussed Bayesian methods in statistical process control. A pair of threshold values on the posterior odds ratio is used to make a three-stage decision about a process: accept without further inspection, adjust (reject) and continue inspecting, or continue inspecting. Forster [1] considered the importance of model selection criteria with a three-way decision: accept, reject or suspend judgment. Goudey [2] discussed three-way statistical inference that supports three possible actions for an environmental manager: act as if there is no problem, act as if there is a problem, or act as if there is not yet sufficient information to allow a decision. Although these studies are about hypothesis testing, they are much in line with the philosophy of three-way decision in rough set theory. Further exploration on such a connection may enrich rough set theory.

3 Conclusion

We present an alternative interpretation of rules in rough set theory based on the notion of three-way decisions. The three-way decision, as expressed by the positive, boundary, and negative rules, reflects more accurately the philosophy and power of rough set theory. It focuses on the actions implied by decision rules, rather than their statistical features. Unlike the existing interpretations, the proposed interpretation consistently explains rules in both the classical model and the probabilistic models. The new interpretation opens up a different avenue of research. One can relate rough set theory to statistical analysis methods involving three-way decision-making. One can also apply rough set theory to problems where three-way decisions are required.

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