Rough Sets, Neighborhood Systems, and Granular Computing

Y.Y. Yao
Department of Computer Science
University of Regina
Regina, Saskatchewan, Canada S4S 0A2
E-mail: yyao@cs.uregina.ca

Abstract

Granulation of a universe involves grouping of similar elements into granules. With granulated views, we deal with approximations of concepts, represented by subsets of the universe, in terms of granules. This paper examines the problem of approximations with respect to various granulations of the universe. The granulation structures used by both rough set theory and neighborhood systems, and the corresponding approximation structures, are studied.

1 Introduction

An underlying idea of granular computing is the use of groups, classes, or clusters of elements called granules [18, 19]. There are many reasons for the study of granular computing. The practical necessity and simplicity in problem solving are perhaps some of the main reasons. When a problem involves incomplete, uncertain, or vague information, it may be difficult to differentiate distinct elements and one is forced to consider granules. Although detailed information may be available, it may be sufficient to use granules in order to have an efficient and practical solution. Very precise solutions may not be required for many practical problems. The use of granules generally leads to simplification of practical problems. It may also happen that the acquisition of precise information is too costly, and coarse-grained information reduces cost. Granular computing will play an important role in the design and implementation of intelligent information systems.

The construction, representation, and interpretation of granules, as well as the utilization of granules for problem solving, are some of the fundamental issues of granular computing. A general framework of granular computing was presented in a recent paper by Zadeh [18] in the context of fuzzy set theory. On the other hand, many researchers investigated specific and more concrete models of granular computing. Lin [2] and Yao [14] studied granular computing using neighborhood systems for the interpretation of granules. Pawlak [5], Polkowski and Skowron [6], and Skowron and Stepaniuk [8] examined granular computing in connection with the theory of rough sets. A salient feature of these studies is that a particular semantics interpretation of granules is defined, and an algorithm for constructing granules is given.

Information granulation depends on the available knowledge. Elements in a granule are drawn together by indistinguishability, similarity, proximity, or functionality [18]. The theories of rough sets and neighborhood systems provide convenient and effective tools for granulation, and deal with some fundamental granulation structures. In the rough set theory, one starts with an equivalence relation. A universe is divided into a family of disjoint subsets. The granulation structure adopted is a partition of the universe. By weakening the requirement of equivalence relations, we can have more general granulation structures such as coverings of the universe. Neighborhood systems provide an even more general granulation structure. For each element of a universe, one associates it with a nonempty family of neighborhood granules, which is called a neighborhood system. It offers a multi-layered granulation of the universe, which is a natural generalization of the single-layered granulation structure used by rough set theory.

With the granulation of universe, one considers elements within a granule as a whole rather than individually [19]. The loss of information through granulation implies that some subsets of the universe can only be approximately described. In theory of rough sets, a pair of lower and upper approximation is typically used. The approximations are expressed in terms of granules according to their overlaps with the set to be approximated. Based on this idea, the main objective of the paper is to study the two related issues of granulation and approximation. The granulation structures used by theories of rough sets and neighborhood systems are analyzed and compared, and the corresponding approximation structures are investigated.

2 Granulations and Approximations

From view of points of rough sets and neighborhood systems, this section examines connections between granulations and approximations.

2.1 Rough sets: granulation by partitions

Let U be a finite and non-empty set called the universe, and let $E \subseteq U \times U$ denote an equivalence relation on U. The pair $apr = (U, E)$ is called an approximation space. The equivalence relation E partitions the set U into disjoint subsets. This partition of the universe is denoted by U/E. The equivalence relation is the available information or knowledge about the objects under consideration. If two elements x, y in U belong to the same equivalence class, we say that x and y are indistinguishable. Each equivalence class may be viewed as a granule consisting of indistinguishable elements, and it is also referred to as an equivalence granule. The granulation structure induced by an equivalence relation is a partition of the universe.

An arbitrary set $X \subseteq U$ may not necessarily be a union of some equivalence classes. This implies that one may not be able to describe X precisely using the equivalence classes of E. In this case, one may characterize X by a pair of lower and upper approximations:

$$\begin{aligned} \underline{apr}(X) &= \bigcup_{[x]_E \subseteq X} [x]_E, \\ \overline{apr}(X) &= \bigcup_{[x]_E \cap X \neq \emptyset} [x]_E, \end{aligned} \tag{1}$$

where

$$[x]_E = \{y \mid xEy\}, \tag{2}$$

is the equivalence class containing x. The lower approximation $\underline{apr}(X)$ is the union of all the equivalence granules which are subsets of X. The upper approximation $\overline{apr}(X)$ is the union of all the equivalence granules which have a non-empty intersection with X.

Lower and upper approximations are dual to each other in the sense:

$$\begin{aligned} &\text{(Ia)} \quad \underline{apr}(X) = (\overline{apr}(X^c))^c, \\ &\text{(Ib)} \quad \overline{apr}(X) = (\underline{apr}(X^c))^c, \end{aligned}$$

where $X^c = U - X$ is the complement of X. The set X lies within its lower and upper approximations:

$$\text{(II)} \quad \underline{apr}(X) \subseteq X \subseteq \overline{apr}(X).$$

Intuitively, lower approximation may be understood as the pessimistic view and the upper approximation the optimistic view in approximating a set by using equivalence granules. One can also verify the following properties:

$$\begin{aligned} &\text{(IIIa)} \quad \underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y), \\ &\text{(IIIb)} \quad \overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y). \end{aligned}$$

The lower (upper) approximation of the intersection (union) of a finite number of sets can be obtained from their lower (upper) approximations. However, we only have:

$$\begin{aligned} &\text{(IVa)} \quad \underline{apr}(X \cup Y) \supseteq \underline{apr}(X) \cup \underline{apr}(Y), \\ &\text{(IVb)} \quad \overline{apr}(X \cap Y) \subseteq \overline{apr}(X) \cap \overline{apr}(Y). \end{aligned}$$

It is impossible to obtain the lower (upper) approximation of the union (intersection) of some sets from their lower (upper) approximations. Additional properties of rough set approximations can be found in Pawlak [4], and Yao and Lin [17].

Equivalence classes of the partition U/E are called the elementary granules. They represent the available information. All knowledge we have about the universe are about these elementary granules, instead of about individual elements. With this interpretation, we also have knowledge about the union of some elementary granules. The empty set $\emptyset$ and the union of one or more elementary sets are usually called definable, observable, measurable, or composed sets. In this study, we call them granules. The set of all granules is denoted $GK(U)$, which is a subset of the power set 2^U. By extending equivalence class of x as given by equation (2) to a subset $X \subseteq U$, we have:

$$[X]_E = \bigcup_{x \in X} [x]_E. \tag{3}$$

Thus, each element of $GK(U)$ may be viewed as the equivalence granule containing a subset of the universe, and the set $GK(U)$ is defined by:

$$GK(U) = \{[X]_E \mid X \subseteq U\}. \tag{4}$$

The set of granules $GK(U)$ is closed under both set intersection and union. It is in fact a σ-algebra of subsets of U generated by the family of equivalence classes U/E.

For an element $G \in GK(U)$, we have:

$$\underline{apr}(G) = G = \overline{apr}(G). \quad (5)$$

For an arbitrary subset $X \subseteq U$, we have the following equivalent definition of rough set approximations:

$$\underline{apr}(X) = \bigcup\{G \mid G \subseteq X, G \in GK(U)\},$$
$$\overline{apr}(X) = \bigcap\{G \mid X \subseteq G, G \in GK(U)\}. \quad (6)$$

This definition offers another interesting interpretation. The lower approximation is the largest granule contained in X, where the upper approximation is the smallest granule containing X. They therefore represent the best approximation of X from below and above using granules.

2.2 Generalized rough sets: granulation by coverings

Granulation of the universe by family of disjoint subsets is a simple and easy to analysis case. One may consider general cases by extending partitions to coverings of the universe, or by extending equivalence relations to arbitrary binary relations [13]. In this section, we use the covering induced by a reflexive binary relation.

Let $R \subseteq U \times U$ be a binary relation on U. For two elements $x, y \in U$, if xRy, we say that y is R-related to x. A binary relation may be more conveniently represented using successor neighborhoods:

$$(x)_R = \{y \in U \mid xRy\}. \quad (7)$$

The successor neighborhood $(x)_R$ consists of all R-related elements of x. When R is an equivalence relation, $(x)_R$ is the equivalence class containing x. When R is a reflexive relation, the family of successor neighborhoods $U/R = \{(x)_R \mid x \in U\}$ is a covering of the universe, namely, $\bigcup_{x \in U}(x)_R = U$. The binary relation R represents the similarity between elements of a universe. It is reasonable to assume that similarity is at least reflexive, but not necessarily symmetric and transitive [9].

For the granulation induced by the covering U/R, rough set approximations can be defined by generalizing equation (1). The equivalence class $[x]_E$ may be replaced by the successor neighborhood $(x)_R$. One of such generalizations is given by [11]:

$$\underline{apr}(X) = \bigcup_{(x)_R \subseteq X} (x)_R,$$
$$\overline{apr}(X) = (\underline{apr}(X^c))^c. \quad (8)$$

In this definition, we generalize the lower approximation and define the upper approximation through duality. In general, $\overline{apr}(X)$ is different from the straightforward generalization $\bigcup_{(x)_R \cap X \neq \emptyset}(x)_R$. While the lower approximation is the union of some successor neighborhoods, the upper approximation cannot be expressed in this way [11].

Similar to the case of partition, we call the elements of a covering elementary granules. The empty set $\emptyset$ or the union of some elementary granules is referred to as a granule. For a subset $X \subset U$, we define:

$$(X)_R = \bigcup_{x \in X} (x)_R, \quad (9)$$

which is the successor neighborhood of X. The set of all such neighborhoods is given by:

$$GK(U) = \{(X)_R \mid X \subseteq U\}. \quad (10)$$

The set $GK(U)$ is only closed under set union. The complemented system:

$$GK^c(U) = \{G^c \mid G \in GK(U)\}, \quad (11)$$

is only closed under set intersection. In fact, $GK^c(U)$ is a closure system [12]. For an element $G \in GK(U)$, i.e., $G^c \in GK^c(U)$, we have:

$$\underline{apr}(G) = G,$$
$$\overline{apr}(G^c) = G^c. \quad (12)$$

In general, $G = \underline{apr}(G) \neq \overline{apr}(G)$ and $\underline{apr}(G^c) \neq \overline{apr}(G^c) = G^c$ for an arbitrary $G \in GK(U)$. By these properties, we refer to the elements of $GK(U)$ as inner definable granules, and the elements of $GK^c(U)$ as outer definable granules [12]. Using these granules, we have another equivalent definition:

$$\underline{apr}(X) = \bigcup\{G \mid G \subseteq X, G \in GK(U)\},$$
$$\overline{apr}(X) = \bigcap\{G \mid X \subseteq G, G \in GK^c(U)\}. \quad (13)$$

The lower approximation is the largest inner definable granule contained in X, and the upper approximation is the smallest outer definable granules containing X. They are related to the definition for the case of partitions, in which $GK(U)$ and $GK^c(U)$ are the same set. For a covering, the set $GK(U) \cap GK^c(U)$ consists of both inner and outer definable granules. Obviously, $\emptyset, U \in GK(U) \cap GK^c(U)$.

The new approximations satisfy properties (I), (II), and (IV). They do not satisfy property (III). Nevertheless, they satisfy a weaker version:

(Va) $\underline{apr}(X \cap Y) \subseteq \underline{apr}(X) \cap \underline{apr}(Y)$,

(Vb) $\overline{apr}(X \cup Y) \supseteq \overline{apr}(X) \cup \overline{apr}(Y)$.

By definition, $\underline{apr}(X \cap Y)$ can be written as a union of some elementary granules. Although both $\underline{apr}(X)$ and $\underline{apr}(Y)$ can be expressed as unions of elementary granules, $\underline{apr}(X) \cap \underline{apr}(Y)$ cannot be so expressed.

2.3 Neighborhood systems: multi-layered granulations

In the theory of rough sets, single-layered granulation structures of the universe are used. The granulated view of the universe is based on a binary relation representing the simplest type of relationships between elements of a universe. Two elements are either related or unrelated. To resolve this difficulty, the notion of neighborhood systems is used to derive more general granulation structures on the universe. Two granulation structures are defined from a neighborhood system. One is a single covering of the universe, and the other is a layered family of coverings of the universe.

The concept of neighborhood systems was originally introduced by Sierpenski and Krieger [10] for the study of Féchet (V)spaces. Lin [1, 2] adopted it for describing relationships between objects in database systems. Yao [14] used the notion for granular computing by focusing on the granulation structures induced by neighborhood systems.

For an element x of a finite universe U, one associates with it a subset $n(x) \subseteq U$ called the neighborhood of x. Intuitively speaking, elements in a neighborhood of an element are somewhat indiscernible or at least not noticeably distinguishable from x. A neighborhood of x may or may not contain x. A neighborhood of x containing x is called a reflexive neighborhood. We are only interested in reflexive neighborhoods of x to accommodate the intuitive interpretation of neighborhoods. A neighborhood system $\mathrm{NS}(x)$ of x is a nonempty family of neighborhoods of x. Distinct neighborhoods of x consist of elements having different types of, or various degrees of, similarity to x. A neighborhood system is reflexive, if every neighborhood in it is reflexive. Let $\mathrm{NS}(U)$ denote the collection of neighborhood systems for all elements in U. It determines a Féchet (V)space, written $(U, \mathrm{NS}(U))$. There is no additional requirements on neighborhood systems.

Neighborhood systems can be used to describe more general types of relationships between elements of a universe [2, 14]. A binary relation can be interpreted in terms of 1-neighborhood systems, in which each neighborhood system contains only one neighborhood [11]. More precisely, the neighborhood system of x is given by $\mathrm{NS}(x) = \{(x)_R\}$. If R is a reflexive relation, one obtains a reflexive neighborhood system which is the covering U/R. If R is an equivalence relation, the successor neighborhood $(x)_R$ is the equivalence class containing x, and the neighborhood system is the partition U/R. With the introduction of multi-neighborhood, we consider various granulations and the corresponding approximations.

A simple method for defining approximations is to construct a covering of the universe by using all neighborhoods in every reflexive neighborhood system:

$$\begin{aligned} C_0 &= \bigcup_{x \in U} \mathrm{NS}(x) \\ &= \{n(x) \mid n(x) \in \mathrm{NS}(x), x \in U\}. \end{aligned} \tag{14}$$

Each granule in C_0 is a neighborhood of an element of the universe. The approximations are defined by:

$$\begin{aligned} \underline{apr}_{C_0}(X) &= \bigcup_{n(x) \subseteq X} n(x), \\ \overline{apr}_{C_0}(X) &= (\underline{apr}(X^c))^c. \end{aligned} \tag{15}$$

A disadvantage of this formulation is that it uses a single-layered granulation structure, and does not make full use of the information provided by neighborhood systems.

In a neighborhood system, different neighborhoods represent different types or degrees of similarity. Such information should be taken into consideration in the approximation. From a neighborhood system of the universe, we may construct a family of coverings of the universe. Instead of using *all* neighborhoods, each covering is obtained by selecting *one* particular neighborhood of each element, i.e.,

$$C = \{n(x), \ldots, n(y), n(z)\}, \tag{16}$$

where $n(x) \in \mathrm{NS}(x), \ldots, n(y) \in \mathrm{NS}(y), n(z) \in \mathrm{NS}(z)$ for $x, \ldots, y, z \in U$. In this way, we transform a neighborhood system into a family of 1-neighborhood systems $FC(U)$. An order relation $\preceq$ on $FC(U)$ can be defined as follows, for $C_1, C_2 \in FC(U)$,

$$C_1 \preceq C_2 \Longleftrightarrow n_{C_1}(x) \subseteq n_{C_2}(x), \quad \text{for all } x \in U. \tag{17}$$

The covering C_1 is finer than C_2, or C_2 is coarser than C_1. For each granule in C_2, one can find a granule in C_1 which is at least as small as the former. It can be verified that $\preceq$ is reflexive, transitive, and anti-symmetric. In other words, $\preceq$ is a partial order, and the set $FC(U)$ is a poset. Thus, we have obtained a family of multi-layered coverings, which in turn produces multi-layered granulations of the universe.

For each covering $C \in FC(U)$, we can define a pair of lower and upper approximations:

$$\begin{aligned} \underline{apr}_C(X) &= \bigcup_{G \in C, G \subseteq X} G, \\ \overline{apr}_C(X) &= (\underline{apr}(X^c))^c. \end{aligned} \quad (18)$$

With the poset $FC(U)$, we obtain multi-layered approximations. Approximations in various layers satisfy the property:

$$C_1 \preceq C_2 \implies \underline{apr}_{C_2}(X) \subseteq \underline{apr}_{C_1}(X), \quad \overline{apr}_{C_1}(X) \subseteq \overline{apr}_{C_2}(X). \quad (19)$$

A finer covering C_1 produces a better approximation than a coarser covering C_2.

In the above formulation, we have transformed general reflexive neighborhood systems into a family of reflexive 1-neighborhood systems. This enables us to apply the results about approximations from the theory of rough sets. Our formulation is indeed based on two basic granulation structures, i.e., partitions and coverings of the universe. They are interpreted by using equivalence and reflexive relations. Consequently, two types of approximations are examined. In the discussion, we focused on only one definition of rough set approximations. There are several different definitions available [11]. The argument presented here can be easily applied to other definitions.

The use of nested sequences of binary relation has also been discussed by many authors. Marek and Rasiowa [3] considered gradual approximations of sets based on a descending sequence of equivalence relations. Pomykala [7] used a sequence of tolerance relations (i.e., reflexive and symmetric relations). Some recent results on this topic were given by Yao [15], and Yao and Lin [16]. The results reported in this paper are more general.

3 Conclusion

The granulation structures adopted in the rough set theory are based on partitions and coverings of a universe, which produce single-layered granulation structures. A set of elementary granules are used to build larger granules that are the set of inner and/or definable granules. In the case of partition, we obtain an σ-algebra representing the set of inner and outer definable granules. In the case of covering, the set of outer definable granules form a closure systems. Every subset of the universe is approximated from below by inner definable granules, and from above by outer definable granules. By using neighborhoods, we can have a family of coverings, which leads to a more general multi-layered granulation structure.

Granulation structures and the corresponding approximation structures introduced in this paper provide a starting point for further study of granulation and approximation. Investigations in this direction may produce interesting and useful results.

References

[1] Lin, T.Y. Neighborhood systems and approximation in relational databases and knowledge bases, *Proceedings of the 4th International Symposium on Methodologies of Intelligent Systems*, 1988.

[2] Lin, T.Y. Granular computing on binary relations I: data mining and neighborhood systems, II: rough set representations and belief functions, in: *Rough Sets in Knowledge Discovery 1*, Polkowski, L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, 107-140, 1998.

[3] Marek, W. and Rasiowa, H. Gradual approximating sets by means of equivalence relations, *Bulletin of Polish Academy of Sciences, Mathematics*, **35**, 233-238, 1987.

[4] Pawlak, Z. Rough sets, *International Journal of Computer and Information Sciences*, **11**, 341-356, 1982.

[5] Pawlak, Z. Granularity of knowledge, indiscernibility and rough sets, *Proceedings of 1998 IEEE International Conference on Fuzzy Systems*, 106-110, 1998.

[6] Polkowski, L. and Skowron, A. Towards adaptive calculus of granules, *Proceedings of 1998 IEEE International Conference on Fuzzy Systems*, 111-116, 1998.

[7] Pomykala, J.A. A remark on the paper by H. Rasiowa and W. Marek: "Gradual approximating sets by means of equivalence relations", *Bulletin of Polish Academy of Sciences, Mathematics*, **36**, 509-512, 1988.

[8] Skowron, A. and Stepaniuk, J. Information granules and approximation spaces, manuscript, 1998.

[9] Slowinski, R. and Vanderpooten, D. Similarity relation as a basis for rough approximations, in: *Advances in Machine Intelligence*

& Soft-Computing, Wang, P.P. (Ed.), Department of Electrical Engineering, Duke University, Durham, North Carolina, 17-33, 1997.

[10] Sierpenski, W. and Krieger, C. *General Topology*, University of Toronto, Toronto, 1956.

[11] Yao, Y.Y. Relational interpretations of neighborhood operators and rough set approximation operators, *Information Sciences*, **111**, 239-259, 1998.

[12] Yao, Y.Y. On generalizing Pawlak approximation operators, in: *Rough Sets and Current Trends in Computing, Lecture Notes in Artificial Intelligence 1424*, Polkowski, L. and Skowron, A. (Eds.), Springer-Verlag, Berlin, 298-307, 1998.

[13] Yao, Y.Y. Generalized rough set models, in: *Rough Sets in Knowledge Discovery 1*, Polkowski, L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, 286-318, 1998.

[14] Yao, Y.Y. Granular computing using neighborhood systems, in: *Advances in Soft Computing: Engineering Design and Manufacturing*, Roy, R., Furuhashi, T., and Chawdhry, P.K. (Eds.), Springer-Verlag, London, 539-553, 1999.

[15] Yao, Y.Y. Stratified rough sets and granular computing, manuscript, 1999.

[16] Yao, Y.Y. and Lin, T.Y. Graded rough set approximations based on nested neighborhood systems, *Proceedings of 5th European Congress on Intelligent Techniques & Soft Computing*, 196-200, 1997.

[17] Yao, Y.Y. and Lin, T.Y. Generalization of rough sets using modal logic, *Intelligent Automation and Soft Computing, an International Journal*, **2**, 103-120, 1996.

[18] Zadeh, L.A. Towards a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, **19**, 111-127, 1997.

[19] Zadeh, L.A. Fuzzy sets and information granularity, in: *Advances in Fuzzy Set Theory and Applications*, Gupta, N., Ragade, R. and Yager, R. (Eds.), North-Holland, Amsterdam, 3-18, 1979.