Application of Matrix Logic to Fuzzy Computing

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Abstract: *In matrix logic, the truth value of a logical expression is represented as a two dimensional vector in the logic vector space. In it logical connectives are defined as matrix operators over logic vectors. Using this approach, we can argue with the evaluation of logical formulas from the point of matrix algebra. We apply this system to fuzzy computing, where we define the correspondence between the matrix operations and composition rules for the fuzzy membership functions. We sketch overall fuzzy inference scheme in terms of matrix logic.*

1 Introduction

Fuzzy computing [2] is important in the practical AI systems. Especially in the diagnosis, control and prediction systems and also in computing with words in intelligent information retrieval systems. In this paper, we apply matrix logic [1] to fuzzy computing. In matrix logic, the truth value of a logical expression is represented as a two dimensional vector in the logic vector space. In it logical connectives are defined as matrix operators over logic vectors. Using this approach, we can argue with the evaluation of logical formulas from the point of matrix algebra. We apply this system to fuzzy computing. In Chapter 2 we briefly introduce matrix logic. In Chapter 3 we define the correspondence between the matrix operations and composition rules for the fuzzy membership functions. We sketch overall fuzzy inference scheme in terms of matrix logic in Chapter 4.

2 Matrix Logic

Matrix logic is developed by August Stern. Using matrix logic, we can make fusion of standard logics and non-standard logics [4]. A standard logic is the crisp logic based on crisp set theory. Fuzzy logic is one of the non-standard logics, and it is based on the fuzzy set theory. Fuzzy logic is thought of one of the many valued logic. The fuzzy inference is one sort of the approximate reasoning. A fuzzy set is the set whose boundary is not defined shaply. In matrix logic, the truth value of a proposition (ground atomic formula) is represented by the logic vector. The logic vector is defined as follows.

$$\begin{array}{ll} \langle p| = (\overline{p}, p) & \text{: called } \textit{bra} \text{ vector} \\ |q\rangle = \begin{pmatrix} \overline{q} \\ q \end{pmatrix} & \text{: called } \textit{ket} \text{ vector} \end{array} \tag{1}$$

where p and q in the left side of (1) are propositions i.e. ground atomic formulas, whereas p and q in the right side represent truth values. In 2-valued logic, if p is true, then $\langle p| = (0,1)$. In many valued logic, $\langle p| = (\overline{p}, p)$. It is usually $\overline{p} + p = 1$. In matrix logic, the inner product and outer product of logic vectors are written as follows.

$$\begin{array}{ll} \langle p|q\rangle = \overline{p} \bullet \overline{q} + p \bullet q & \text{: inner product} \\ |p\rangle\langle q| = \begin{pmatrix} \overline{p}\bullet\overline{q} & \overline{p}\bullet q \\ p\bullet\overline{q} & p\bullet q \end{pmatrix} & \text{: outer product} \end{array} \tag{2}$$

In the Stern's matrix logic, $\bullet$ means the standard scaler multiplication. We later change this $\bullet$-operator in order to define composition rules for fuzzy membership functions. The matrix logic operator L(x,y) is a 2×2 matrix such that

$$\begin{aligned} L(x,y) = \langle x|L|y\rangle = \begin{pmatrix} \overline{x} & x \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \overline{y} \\ y \end{pmatrix} \\ = \overline{p} \bullet L_{11} \bullet \overline{q} + \overline{p} \bullet L_{12} \bullet q + p \bullet \mathrm{L}_{21} \bullet \overline{q} + p \bullet \mathrm{L}_{22} \bullet q \end{aligned} \tag{3}$$

For example, logical connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$ are defined as the matrix logic operators as follows.

$$\wedge = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \vee = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{4}$$

$$\neg = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rightarrow = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5}$$

Many other logic operators are defined in matrix logic. The most interesting and useful thing in matrix logic is such configuration of logic operators. Using matrix logic, we can more mathematically analyse many features of symbolic logics in the unified form.

3 Defining Composition Rules via Matrix Logic

Many researchers have defined conposition rules for fuzzy sets (fuzzy membership functions). In general, to define composition rules, these rules are required to satisfy certan properties. These properties are commutativity, associativity, distributivity, idempotency, and de Morgan's law. The composition rules using *min* and *max* operators are the most well known operators that satisfy these properties. In this chapter, we define the other composition rules which satisfy these properties in terms of matrix logic. First, we define the correspondence between membership value $\mu_A(x)$ for x in a fuzzy set A and the inner product of the logic vector as follows. This is possible because the fuzzy membership function is interpreted to be truth functional [3].

Definition1

$$\mu_A(x) = \langle A(x)|1\rangle = a_x \tag{6}$$
$$\mu_{\neg A}(x) = \langle 1|\neg|A(x)\rangle = \overline{a}_x = 1 - a_x \tag{7}$$

Here $A(x)$ means the fuzzy proposition 'x *is* A' and $\langle A(x)| = (\overline{a}_x \;\; a_x), \langle 1| = (0 \;\; 1)$. Next, we change the $\bullet$-operator as follows.

Definition2

For $a_x = \mu_A(x) \in [0,1]$, $b_x = \mu_B(x) \in [0,1]$,

$$a_x \bullet b_x = \begin{cases} a_x & iff \quad A = B \\ a_x b_x & otherwise \end{cases} \tag{8}$$

In the sequel, we simply write a, b, c for $\mu_A(x), \mu_B(x)$ and $\mu_C(x)$ respectively. ¿From this definition, the $\bullet$-operator has the following properties.

$$a \bullet a = a \tag{9}$$
$$a \bullet b = b \bullet a \tag{10}$$
$$a \bullet (b \bullet c) = (a \bullet b) \bullet c \tag{11}$$

That is, the $\bullet$-operator has the properties of idempotency, commutativity and associativity. However, it is not distributive in general. This is seen in the case of $(a+b)\bullet(a+b)$. From the definition, $(a+b)\bullet(a+b)$ should be eqaul to $(a+b)$. However, if the $\bullet$-operator is assumed distributive, $(a+b) \bullet (a+b) = a \bullet a + 2(a \bullet b) + b \bullet b = a + 2(a \bullet b) + b \neq a + b$. In the case of $(1-a) \bullet (1-a)$, we can see $(1-a)(1-a) = 1 - 2a + a \bullet a = 1 - a = (1-a) \bullet (1-a)$. Next, we define composition rules for fuzzy set A and fuzzy set B as follows.

Definition3

$$\begin{aligned}
A \cap B \quad & \mu_{A\cap B}(x) \\
& = \langle A(x)| \wedge |B(x)\rangle = \mu_A(x) \bullet \mu_B(x) && (12) \\
A \cup B \quad & \mu_{A\cup B}(x) = \langle A(x)| \vee |B(x)\rangle \\
& = \mu_A(x) + \mu_B(x) - \mu_A(x) \bullet \mu_B(x) && (13) \\
\neg A \quad & \mu_{\neg A}(x) = \langle 1|\neg|A(x)\rangle = 1 - \mu_A(x) && (14) \\
A \rightarrow B \quad & \mu_{A\rightarrow B}(x) = \langle A| \rightarrow |B\rangle \\
& = 1 - \mu_A(x) + \mu_A(x) \bullet \mu_B(x) \\
& = \mu_{\neg A}(x) + \mu_A(x) \bullet \mu_B(x) && (15)
\end{aligned}$$

The composition rules defined above satisfy the following axioms that are required to be fulfilled in general.

Axioms for fuzzy composition rules

$$\begin{aligned}
\otimes: \quad & x \otimes 1 = x, x \otimes 0 = 0 \\
& x \otimes y = y \otimes x \\
& x \leq y \Rightarrow x \otimes z \leq y \otimes z \\
& x \otimes (y \otimes z) = (x \otimes y) \otimes z && (16) \\
\oplus: \quad & x \oplus 1 = 1, x \oplus 0 = x \\
& x \oplus y = y \oplus x \\
& x \leq y \Rightarrow x \oplus z \leq y \oplus z \\
& x \oplus (y \oplus z) = (x \oplus y) \oplus z && (17) \\
\neg: \quad & \neg 0 = 1, \neg(\neg x) = x \\
& x < y \Rightarrow \neg x > \neg y && (18)
\end{aligned}$$

The composition rules given in (12), (13) and (14) satisfy (16), (17) and (18) respectively. We can show that the composition rules defined above satisfy commutativity, associativity, distributivity, idempotency and de Morgan's law. However, if we interprete $\bullet$-operator to be the ordinary algebraic multiplication, these rules fail to satify distributivity, idempotency and de Morgan's law. We prove here that our formulation satisfies the distributive law $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$ and $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$. The proof of the latter is as follows.

$$\mu_{A\cup(B\cap C)} = a + b \bullet c - a \bullet b \bullet c \tag{19}$$

$$\begin{aligned}
\mu_{(A\cup B)\cap(A\cup C)} &= (a + b - a \bullet b) \bullet (a + c - a \bullet c) \\
&= (a + b - a \bullet b)(a + c - a \bullet c) \\
&= aa + ba - (a \bullet b)a \\
&+ac + bc - (a \bullet b)c \\
&-a(a \bullet c) - b(a \bullet c) + (a \bullet b)(a \bullet c) \\
&= a + b \bullet c - a \bullet b \bullet c && (20)
\end{aligned}$$

In the proof, we assumed $A \cup B \neq A \cup C$ and we used Definition1 of the $\bullet$-operator. Namely, $aa = a \bullet a = a$, $bc = b \bullet c$, etc.. In the case of $A \cup B = A \cup C$, this means B = C. The proof of this case is omitted here but it is easily obtained in the similar fashion. Similary we can prove the other equivalence relations. We notice here that from (15), (9) and the definition 1, $\mu_{A\rightarrow A}(x) = 1$ always holds. Similary, $\mu_{\neg A\cup A}(x) = 1$ always holds because $\mu_{\neg A\cup A} = \mu_{\neg A} + \mu_A - \mu_{\neg A}\mu_A = 1$ from (13). Hence, both $A \rightarrow A$ and $\neg A \vee A$ are tautologies and always true. From Eqn. (12), $\neg A \wedge A$ is alway false. This means that both the law of excluded middle and the law of contradiction hold in our formulation. That is the extreme characteristic of our composition rules.

4 Fuzzy Inference Process

The overall fuzzy inference process consists of (a) fuzzification, (b) fuzzy inference and (c) defuzzification. The fuzzy inference is performed by applying the fuzzy modus ponens rule dipicted in (21). In the modus pones rule, we infer B^* from the given fact A^* and the fuzzy rule $A \to B$. In the fuzzy inference system, this fuzzy inference is mechanically carried out by performing the certain fuzzy unification between A^* and A and then obtain the conclusion B^* determining fuzziness of B^* by the defined fuzzy composition rules. Moreover, the grade of the truth or certainty of this inference would be calculated. The Eqn.(22) shows this overall fuzzy inference process.

$$\frac{\begin{array}{lcl} A & \to & B \\ A^* & & \end{array}}{\qquad\qquad B^*} \tag{21}$$

$$\frac{\begin{array}{lll} A^* \sim A & A \to B & B \sim B^* \\ A^* \leftarrow x & & \end{array}}{\qquad\qquad B^* \to y} \tag{22}$$

In (22), $A^* \leftarrow x$ denotes the fuzzification process that fuzzifies x to be A^*. $A^* \sim A$ denotes the fuzzy unification process between A^* and A. $B \sim B^*$ denotes the fuzzy identification process that identifies B^* with B. $B^* \to y$ denotes the defuzzyfication process that defuzzifies B^* to be y. In this fuzzy inference process (22), we take account of the given membership functions μ_{A^*}, μ_A and μ_B for the calculation of the fuzziness of the conclusion B^*. The grade of the truth of the fuzzy inference (22) is calculated by the following formula $m(A^*, A, B)$.

$$\begin{aligned} m(A^*, A, B) &\equiv \|(A^* \wedge (A \to B)) \to B^*\| \\ &= \overline{\langle a^*|1\rangle\langle a| \to |b\rangle\langle 0|b^*\rangle} \\ &= 1 - \langle a^*| \begin{pmatrix} 0 & 0 \\ \overline{a} + a \bullet b & 0 \end{pmatrix} |b^*\rangle \\ &= 1 - \langle a^*|\backslash|b^*\rangle \\ &= 1 - a^* \bullet (\overline{a} + a \bullet b) \bullet \overline{b^*} \end{aligned} \tag{23}$$

In the above formula, \ is called inhibition operator. It is obtained by calculating $|1\rangle\langle a| \to |b\rangle\langle 0|$. Such a capability of reduction by performing the inner product of logic vectors and the matrix multiplication is the most advantegeous feature in matrix logic. In (23), the given data are the values of a^*, a and b. The value of $\overline{b^*}$, namely, b^* is not given. Therefore we have to determine the value of b^* as the value of a certain function of a^*, a and b. How should we do it? Although this function could be arbitrary selectable, we put this function,(denoted by β^*) such that

$$\beta^*(x^*, x, y) = x^* \bullet (\overline{x} + x \bullet y) \tag{24}$$

where we have renamed variables a^*, a and b in (24) to x^*, x and y respectively. Using this β^*-function, the Eqn. (23) is rewritten as follows.

$$m(A^*, B, C) = 1 - \beta^*(a^*, a, b) \bullet \overline{\beta^*(a^*, a, b)} \tag{25}$$

Since $1 - \beta^* \bullet \overline{\beta^*} = 1$, we can say that the β^*-function given by (24) maximizes $m(A^*, A, B)$ with the values $x^* = a^*, x = a$ and $y = b$. This is the reason why we determined the value of b^* in (23) as $b^* = \beta^*(a^*, a, b)$. The table 1 shows values of the function (24) for the special values of x^*, x and y.

x^*	x	y	$\beta^* = x^*(\overline{x} + xy)$
a^*	a	a	a^*
a^*	1	b	a^*b
a^*	a^*	b	a^*b
a^*	a	b	$a^*(1 - a + ab)$

Table 1: The values of β^*

We illustrate a simple example here. Assume that the fuzzy inference rule *'if x is tall, then x is heavy'* and a fact *'John is very tall'* are given as follows.

$$tall(x) : \alpha_x \to heavy(x) : \beta_x \tag{26}$$

$$tall(John) : very_{John} \tag{27}$$

Here α_x and β_x are fuzzy membership functions $\mu_{tall}(x)$ and $\mu_{heavy}(x)$ respectively. $very_{John}$ is the value of the membership function $\mu_{very.tall}(x)$ for $x = John$. Then from the fuzzy modus ponens rule (21), we can derive $heavy(John) : very_{John}(1 - \alpha_{John} + \alpha_{John}\beta_{John})$. If $\alpha_x = \beta_x$ is assumed, then this conclusion becomes $heavy(John) : very_{John}$ from the first entry in Table 1. Namely, *'John is very heavy'* is derived. If $\alpha_x \neq \beta_x$, the grade of heavyness β^* of John *heavy(John)* is given by $very_{John}(1 - \alpha_{John} + \alpha_{John}\beta_{John})$. Let β_x be $\beta_x = 0.9\alpha_x$. Then if $very_{John} = \mu_{very.tall}(John) = 0.8$ and $\alpha_{John} = \mu_{tall}(John) = 1$, then $heavy(John) : 0.72very$ is obtained. We can interprete $heavy(John) : 0.72very$ as *'John is fairly heavy'* by applying the appropriate linguistic translation rule for the predicate $tall(x) : \mu_x$.

5 Concluding Remarks

The author has described an application of matrix logic to fuzzy computing. The remaining subject to be done is to make further investigation of the described method through the applications to the real problems. Another subject is to apply matrix logic to the granulation process in the granular computing. Since the fuzzy computing is one sort of the granular computing, the method described in this paper would contribute to the research for the granular computing.

References

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