Rough Set Approximations in Formal Concept Analysis

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Abstract. A basic notion shared by rough set analysis and formal concept analysis is the definability of a set of objects based on a set of properties. The two theories can be compared, combined and applied to each other based on definability. In this paper, the notion of rough set approximations is introduced into formal concept analysis. Rough set approximations are defined by using a system of definable sets. The similar idea can be used in formal concept analysis. The families of the sets of objects and the sets of properties established in formal concept analysis are viewed as two systems of definable sets. The approximation operators are then formulated with respect to the systems. Two types of approximation operators, with respect to lattice-theoretic and set-theoretic interpretations, are studied. The results provide a better understanding of data analysis using rough set analysis and formal concept analysis.

1 Introduction

Definability deals with whether and how a set can be defined in order to be analyzed and computed [38]. A comparative examination of rough set analysis and formal concept analysis shows that each of them deals with a particular type of definability. While formal concept analysis focuses on sets of objects that can be defined by conjunctions of properties, rough set analysis focuses on disjunction of properties [33]. The common notion of definability links the two theories together. One can immediately adopt ideas from one to the other [33, 34]. On the one hand, the notions of formal concepts and formal concept lattices can be introduced into rough set analysis by considering different types of formal concepts [34]. On the other hand, rough set approximation operators can be introduced into formal concept analysis by considering a different type of definability [8, 35]. The combination of the two theories would produce new tools for data analysis.

An underlying notion of rough set analysis is the indiscernibility of objects [12, 13]. By modelling indiscernibility as an equivalence relation, one can partition a finite universe of objects into a family of pair-wise disjoint subsets called a partition. The partition provides a granulated view of the universe. An equivalence class is considered as a whole, instead of many individuals, and is viewed as an elementary definable subset. In other words, one can only observe, measure, or characterize the equivalence classes. The empty set and unions of equivalence classes are also treated as definable subsets. In general, the system of such definable subsets is only a proper subset of the power set of the universe. Consequently, an arbitrary subset of universe may not necessarily be definable. It can be approximated from below and above by a pair of maximal and minimal definable subsets.

Under the rough set approximation, there is a close connection between definability and approximation. A definable set of the universe of objects must have the same approximations [2]. That is, a set of objects is definable if and only if its lower approximation equals to its upper approximation.

Formal concept analysis is developed based on a formal context given by a binary relation between a set of objects and a set of properties. From a formal context, one can construct (objects, properties) pairs known as the formal concepts [6, 22]. The set of objects of a formal concept is referred to as the extension, and the set of properties as the intension. They uniquely determine each other. The family of all formal concepts is a complete lattice. The extension of a formal concept can be viewed as a definable set of objects, although in a sense different from that of rough set analysis [33, 34]. In fact, the extension of a formal concept is a set of indiscernible objects with respect to the intension. Based on the properties in the intension, all objects in the extension cannot be distinguished. Furthermore, all objects in the extension share all the properties in the intension. The collection of all the extensions, sets of objects, can be considered as a different system of definable sets [35]. An arbitrary set of objects may not be an extension of a formal concept. The sets of objects that are not extensions of formal concepts are regarded as undefinable sets. Therefore, in formal concept analysis, a different type of definability is proposed.

Saquer and Deogun proposed to approximate a set of objects, a set of properties, and a pair of a set of objects and a set of properties, based on a formal concept lattice [16, 17]. Hu *et al.* proposed a method to approximate a set of objects and a set of properties by using join- and meet-irreducible formal concepts with respect to set-theoretic operations [8]. However, their formulations are slightly flawed and fail to achieve such a goal. It stems from a mixed-up of the lattice-theoretic operators and set-theoretic operators. To avoid their limitation, a clear separation of two types of approximations is needed. In this paper, we propose a framework to examine the issues of rough set approximations within formal concept analysis. We concentrate on the interpretations and formulations of various notions. Two systems are examined for the definitions of approximations, the formal concept lattice and the system of extensions of all formal concepts.

The rest of the paper is organized as follows. In Section 2, we discuss three formulations of rough set approximations, subsystem based formulation, granule based formulation and element based formulation. In Section 3, formal concept analysis is reviewed. In Section 4, we apply the notion of rough set approximations into formal concept analysis. Two systems of definable sets are established. Based on each system, different definitions of approximations are examined. Section 5 discusses the existing studies and investigates their differences and connections from the viewpoint of approximations.

2 Rough Set Approximations

The rough set theory is an extension of classical set theory with two additional approximation operators [28]. It is a useful theory and tool for data analysis. Various formulations of rough set approximations have been proposed and studied [30–32]. In this section, we review the subsystem based formulation, granule based formulation and element based formulation, respectively. In the subsystem based formulation, a subsystem of the power set of a universe is first constructed and the approximation operators are then defined using the subsystem. In the granule based formulation, equivalence classes are considered as the elementary definable sets, and approximations can be defined directly by using equivalence classes. In the element based formulation, the individual objects in the equivalence classes are used to calculate approximations of a set of objects.

Suppose U is a finite and nonempty universe of objects. Let $E \subseteq U \times U$ be an equivalence relation on U. The equivalence relation divides the universe into a family of pair-wise disjoint subsets, called the partition of the universe and denoted by U/E. The pair $apr = (U, E)$ is referred to as an approximation space.

An approximation space induces a granulated view of the universe. For an object $x \in U$, the equivalence class containing x is given by:

$$[x]_E = \{y \mid xEy\}. \tag{1}$$

Objects in $[x]_E$ are indistinguishable from x. One is therefore forced to consider $[x]_E$ as a whole. In other words, under an equivalence relation, equivalence classes are the smallest non-empty observable, measurable, or definable subsets of U. By extending the definability of equivalence classes, we assume that the empty set and unions of some equivalence classes are definable. The family of definable subsets contains the empty set $\emptyset$ and is closed under set complement, intersection and union. It is an σ-algebra $\sigma(U/E) \subseteq 2^U$ with basis U/E, where 2^U is the power set of U.

A set of objects not in $\sigma(U/E)$ is said to be undefinable. An undefinable set must be approximated from below and above by a pair of definable sets.

Definition 1. (Subsystem based definition) *In an approximation space $apr = (U, E)$, a pair of approximation operators, $\underline{apr}$, $\overline{apr} : 2^U \longrightarrow 2^U$, is defined by:*

$$\begin{aligned}\underline{apr}(A) &= \bigcup\{X \mid X \in \sigma(U/E), X \subseteq A\},\\ \overline{apr}(A) &= \bigcap\{X \mid X \in \sigma(U/E), A \subseteq X\}.\end{aligned} \tag{2}$$

The lower approximation $\underline{apr}(A) \in \sigma(U/E)$ is the greatest definable set contained in A, and the upper approximation $\overline{apr}(A) \in \sigma(U/E)$ is the least definable set containing A.

Alternatively, the approximation operators can also be defined by using equivalence classes.

Definition 2. (Granule based definition) *In an approximation space* $apr = (U, E)$*, a pair of approximation operators,* $\underline{apr}$*,* $\overline{apr} : 2^U \longrightarrow 2^U$*, is defined by:*

$$\begin{aligned} \underline{apr}(A) &= \bigcup\{[x]_E \mid [x]_E \in U/E, [x]_E \subseteq A\}, \\ \overline{apr}(A) &= \bigcup\{[x]_E \mid [x]_E \in U/E, A \cap [x]_E \neq \emptyset\}. \end{aligned} \tag{3}$$

The lower approximation is the union of equivalence classes that are subsets of A, and the upper approximation is the union of equivalence classes that have a non-empty intersection with A.

The element based definition is another way to define the lower and upper approximations of a set of objects.

Definition 3. (Element based definition) *In an approximation space* $apr = (U, E)$*, a pair of approximation operators,* $\underline{apr}$*,* $\overline{apr} : 2^U \longrightarrow 2^U$*, is defined by:*

$$\begin{aligned} \underline{apr}(A) &= \{x \mid x \in U, [x]_E \subseteq A\}, \\ \overline{apr}(A) &= \{x \mid x \in U, A \cap [x]_E \neq \emptyset\}. \end{aligned} \tag{4}$$

The lower approximation is the set of objects whose equivalence classes are subsets of A. The upper approximation is the set of objects whose equivalence classes have non-empty intersections with A.

The three formulations are equivalent, but with different forms and interpretations [32]. The lower and upper approximation operators have the following properties: for sets of objects A, A_1 and A_2,

$$\begin{aligned}
\text{(i).}\quad & \underline{apr}(A) = (\overline{apr}(A^c))^c, \\
& \overline{apr}(A) = (\underline{apr}(A^c))^c; \\
\text{(ii).}\quad & \underline{apr}(A_1 \cap A_2) = \underline{apr}(A_1) \cap \underline{apr}(A_2), \\
& \overline{apr}(A_1 \cup A_2) = \overline{apr}(A_1) \cup \overline{apr}(A_2); \\
\text{(iii).}\quad & \underline{apr}(A) \subseteq A \subseteq \overline{apr}(A); \\
\text{(iv).}\quad & \underline{apr}(\underline{apr}(A)) = \underline{apr}(A), \\
& \overline{apr}(\overline{apr}(A)) = \overline{apr}(A); \\
\text{(v).}\quad & \underline{apr}(\overline{apr}(A)) = \overline{apr}(A), \\
& \overline{apr}(\underline{apr}(A)) = \underline{apr}(A).
\end{aligned}$$

Property (i) states that the approximation operators are dual operators with respect to set complement ^{c}. Property (ii) states that the lower approximation operator is distributive over set intersection $\cap$, and the upper approximation operator is distributive over set union $\cup$. By property (iii), a set lies within its lower and upper approximations. Properties (iv) and (v) deal with the compositions of lower and upper approximation operators. The result of the composition of a sequence of lower and upper approximation operators is the same as the application of the approximation operator closest to A.

As shown by the following theorem, the approximation operators truthfully reflect the intuitive understanding of the notion of definability [12, 35].

Theorem 1. *In an approximation space $apr = (U, E)$, for a set of objects A, $\underline{apr}(A) = \overline{apr}(A)$ if and only if $A \in \sigma(U/E)$.*

An important implication of the theorem is that for an undefinable set $A \subseteq U$, we have $\underline{apr}(A) \neq \overline{apr}(A)$. In fact, $\underline{apr}(A)$ is a proper subset of $\overline{apr}(A)$, namely, $\underline{apr}(A) \subset \overline{apr}(A)$.

The basic ideas of subsystem based formulation can be generalized by considering different subsystems that represent different types of definability [35]. The granule based formulation and element based formulation can also be generalized by using different types of definable granules [29, 32, 37].

3 Formal Concept Analysis

Formal concept analysis deals with visual presentation and analysis of data [6, 22]. It focuses on the definability of a set of objects based on a set of properties, and vice versa.

Let U and V be any two finite sets. Elements of U are called objects, and elements of V are called properties. The relationships between objects and properties are described by a binary relation R between U and V, which is a subset of the Cartesian product $U \times V$. For a pair of elements $x \in U$ and $y \in V$, if $(x, y) \in R$, written as xRy, we say that x has the property y, or the property y is possessed by object x. The triplet (U, V, R) is called a formal context. By the terminology of rough set analysis, a formal context is in fact a binary information table.

Based on the binary relation, we associate a set of properties to an object. An object $x \in U$ has the set of properties:

$$xR = \{y \in V \mid xRy\} \subseteq V. \tag{5}$$

Similarly, a property y is possessed by the set of objects:

$$Ry = \{x \in U \mid xRy\} \subseteq U. \tag{6}$$

By extending these notations, we can establish relationships between sets of objects and sets of properties. This leads to two operators, one from 2^U to 2^V and the other from 2^V to 2^U.

Definition 4. *Suppose (U, V, R) is a formal context. For a set of objects $A \subseteq U$, we associate it with a set of properties:*

$$\begin{aligned} A^* &= \{y \in V \mid \forall x \in U(x \in A \Longrightarrow xRy)\} \\ &= \{y \in V \mid A \subseteq Ry\} \\ &= \bigcap_{x \in A} xR. \end{aligned} \tag{7}$$

For a set of properties $B \subseteq V$, we associate it with a set of objects:

$$\begin{aligned} B^* &= \{x \in U \mid \forall y \in V(y \in B \Longrightarrow xRy)\} \\ &= \{x \in U \mid B \subseteq xR\} \\ &= \bigcap_{y \in B} Ry. \end{aligned} \tag{8}$$

For simplicity, the same symbol is used for both operators. The actual role of the operators can be easily seen from the context.

By definition, $\{x\}^* = xR$ is the set of properties possessed by x, and $\{y\}^* = Ry$ is the set of objects having property y. For a set of objects A, A^* is the *maximal* set of properties shared by *all* objects in A. For a set of properties B, B^* is the *maximal* set of objects that have *all* properties in B.

The operators * have the following properties [6, 22]: for $A, A_1, A_2 \subseteq U$ and $B, B_1, B_2 \subseteq V$,

$$\begin{array}{ll}
(1). & A_1 \subseteq A_2 \Longrightarrow A_1^* \supseteq A_2^*, \\
 & B_1 \subseteq B_2 \Longrightarrow B_1^* \supseteq B_2^*, \\
(2). & A \subseteq A^{**}, \\
 & B \subseteq B^{**}, \\
(3). & A^{***} = A^*, \\
 & B^{***} = B^*, \\
(4). & (A_1 \cup A_2)^* = A_1^* \cap A_2^*, \\
 & (B_1 \cup B_2)^* = B_1^* \cap B_2^*.
\end{array}$$

In formal concept analysis, one is interested in a pair of a set of objects and a set of properties that uniquely define each other. More specifically, for $(A, B) = (B^*, A^*)$, we have [33]:

$$\begin{aligned}
x \in A &\Longleftrightarrow x \in B^* \\
&\Longleftrightarrow B \subseteq xR \\
&\Longleftrightarrow \bigwedge_{y \in B} xRy; \\
\bigwedge_{x \in A} xRy &\Longleftrightarrow A \subseteq Ry \\
&\Longleftrightarrow y \in A^* \\
&\Longleftrightarrow y \in B.
\end{aligned} \tag{9}$$

That is, the set of objects A is defined based on the set of properties B, and vice versa. This type of definability leads to the introduction of the notion of formal concepts [6, 22].

Definition 5. *A pair (A, B), $A \subseteq U$, $B \subseteq V$, is called a formal concept of the context (U, V, R), if $A = B^*$ and $B = A^*$. Furthermore, $extent(A, B) = A$ is called the extension of the concept, and $intent(A, B) = B$ is called the intension of the concept.*

Definition 6. *For an object x, the pair $(\{x\}^{**}, \{x\}^*)$ is a formal concept and called an object concept. For a property y, the pair $(\{y\}^*, \{y\}^{**})$ is a formal concept and called a property concept.*

The set of all formal concepts forms a complete lattice called a concept lattice, denoted by $L(U, V, R)$ or simply L. The meet and join of the lattice are characterized by the following basic theorem of concept lattices [6, 22].

Theorem 2. *The formal concept lattice L is a complete lattice in which the meet and join are given by:*

$$\bigwedge_{t\in T}(A_t, B_t) = (\bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)^{**}),$$
$$\bigvee_{t\in T}(A_t, B_t) = ((\bigcup_{t\in T} A_t)^{**}, \bigcap_{t\in T} B_t). \tag{10}$$

where T is an index set and for every $t \in T$, (A_t, B_t) is a formal concept.

The order relation of the lattice can be defined based on the set inclusion relation [6, 22].

Definition 7. *For two formal concepts (A_1, B_1) and (A_2, B_2), (A_1, B_1) is a sub-concept of (A_2, B_2), written $(A_1, B_1) \preceq (A_2, B_2)$, and (A_2, B_2) is a super-concept of (A_1, B_1), if and only if $A_1 \subseteq A_2$, or equivalently, if and only if $B_2 \subseteq B_1$.*

A more general (specific) concept is characterized by a larger (smaller) set of objects that share a smaller (larger) set of properties.

The lattice-theoretic operators of meet ($\wedge$) and join ($\vee$) of the concept lattice are defined based on the set-theoretic operators of intersection ($\cap$), union ($\cup$) and the operators *. However, they are not the same. An intersection of extensions (intensions) of a family of formal concepts is the extension (intension) of a formal concept. A union of extensions (intensions) of a family of formal concepts is not necessarily the extension (intension) of a formal concept.

Example 1. The ideas of formal concept analysis can be illustrated by an example taken from [35]. Table 1 gives a formal context, where the meaning of each property is given as follows: a: needs water to live; b: lives in water; c: lives on land; d: needs chlorophyll to produce food; e: two seed leaves; f: one seed leaf; g: can move around; h: has limbs; i: suckles its offspring. Figure 1 gives the corresponding concept lattice. Consider two formal concepts $(\{3, 6\}, \{a, b, c\})$ and $(\{5, 6, 7, 8\}, \{a, d\})$. Their meet is the formal concept:

$$(\{3, 6\} \cap \{5, 6, 7, 8\}, (\{a, b, c\} \cup \{a, d\})^{**}) = (\{6\}, \{a, b, c, d, f\}),$$

and their join is the formal concept:

$$((\{3, 6\} \cup \{5, 6, 7, 8\})^{**}, \{a, b, c\} \cap \{a, d\}) = (\{1, 2, 3, 4, 5, 6, 7, 8\}, \{a\}).$$

The intersection of extensions of two concepts is the extension of their meet, and the intersection of the intensions is the intension of their join. On the other hand, the union of extensions of the two concepts is $\{3, 5, 6, 7, 8\}$, which is not the extension of any formal concept. The union of the intensions is $\{a, b, c, d\}$, which is not the intension of any formal concept.

	a	b	c	d	e	f	g	h	i
1. Leech	×	×					×		
2. Bream	×	×					×	×	
3. Frog	×	×	×				×	×	
4. Dog	×		×				×	×	×
5. Spike-weed	×	×		×		×			
6. Reed	×	×	×	×		×			
7. Bean	×		×	×	×				
8. Maize	×		×	×		×			

Table 1. A formal context taken from [6]

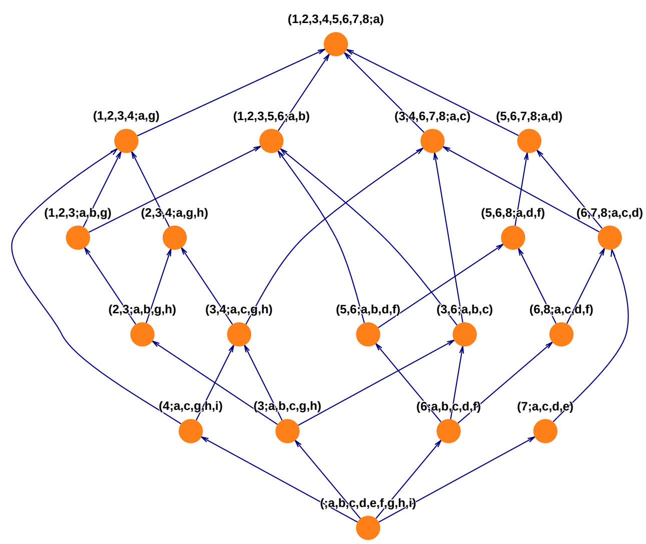

Fig. 1. Concept lattice for the context of Table 1, produced by “Formal Concept Calculator” (developed by Sören Auer, http://www.advis.de/soeren/fca/).

4 Approximations in Formal Concept Analysis

A formal concept is a pair of a definable set of objects and a definable set of properties, which uniquely determine each other. The concept lattice is the family of all concepts with respect to a formal context. Given an arbitrary subset of the universe of objects, it may not be the extension of a formal concept. The set can therefore be viewed as an undefinable set of objects. Following rough sets analysis, such a subset of the universe of objects can be approximated by definable sets of objects, namely, the extensions of formal concepts.

4.1 Approximations based on lattice-theoretic operators

One can develop the approximation operators similar to the subsystem based formulation of rough set analysis. The concept lattice is used as the system of definable concepts, and lattice-theoretic operators are used to define approximation operators.

For a set of objects $A \subseteq U$, suppose we want to approximate it by the extensions of a pair of formal concepts in the concept lattice. We can extend Definition 1 to achieve this goal. In equation (2), set-theoretic operators $\cap$ and $\cup$ are replaced by lattice-theoretic operators $\wedge$ and $\vee$, the subsystem $\sigma(U/E)$ by lattice L, and definable sets of objects by extensions of formal concepts. The extensions of the resulting two concepts are the approximations of A.

Definition 8. (Lattice-theoretic definition) *For a set of objects $A \subseteq U$, its lower and upper approximations are defined by:*

$$\begin{aligned}
\underline{lapr}(A) &= extent(\bigvee\{(X,Y) \mid (X,Y) \in L, X \subseteq A\}) \\
&= (\bigcup\{X \mid (X,Y) \in L, X \subseteq A\})^{**}, \\
\overline{lapr}(A) &= extent(\bigwedge\{(X,Y) \mid (X,Y) \in L, A \subseteq X\}) \\
&= \bigcap\{X \mid (X,Y) \in L, A \subseteq X\}.
\end{aligned} \tag{11}$$

The lower approximation of a set of objects A is the extension of the formal concept $(\underline{lapr}(A), (\underline{lapr}(A))^*)$, and the upper approximation is the extension of the formal concept $(\overline{lapr}(A), (\overline{lapr}(A))^*)$. The concept $(\underline{lapr}(A), (\underline{lapr}(A))^*)$ is the supremum of those concepts whose extensions are subsets of A, and $(\overline{lapr}(A), (\overline{lapr}(A))^*)$ is the infimum of those concepts whose extensions are supersets of A.

For a formal concept (X, Y), X^c may not necessarily be the extension of a formal concept. The concept lattice in general is not a complemented lattice. The approximation operators $\underline{lapr}$ and $\overline{lapr}$ are not necessarily dual operators.

Recall that the intersection of extensions is the extension of a concept, but the union of extensions may not be the extension of a concept. It follows that $(\overline{lapr}(A), (\overline{lapr}(A))^*)$ is the smallest concept whose extension is a superset of A. However, the concept $(\underline{lapr}(A), (\underline{lapr}(A))^*)$ may not be the largest concept whose extension is a subset of A. It may happen that $A \subseteq \underline{lapr}(A)$. That is, the lower approximation of A may not be a subset of A. The new approximation operators do not satisfy properties (i), (ii) and (iii).

With respect to property (ii), they only satisfy a weak version known as monotonicity with respect to set inclusion:

$$\begin{aligned} \text{(vi).} \quad & A_1 \subseteq A_2 \Longrightarrow \underline{lapr}(A_1) \subseteq \underline{lapr}(A_2), \\ & A_1 \subseteq A_2 \Longrightarrow \overline{lapr}(A_1) \subseteq \overline{lapr}(A_2). \end{aligned}$$

The following weak versions of property (iii) are satisfied:

$$\begin{aligned} \text{(vii).} \quad & \underline{lapr}(A) \subseteq \overline{lapr}(A), \\ \text{(viii).} \quad & A \subseteq \overline{lapr}(A). \end{aligned}$$

Both $\underline{lapr}(A)$ and $\overline{lapr}(A)$ are extensions of formal concepts. It follows that the operators $\underline{lapr}$ and $\overline{lapr}$ satisfy properties (iv) and (v).

Example 2. Given the concept lattice in Figure 1, consider a set of objects $A = \{3, 5, 6\}$. The family of subsets of A that are extensions of concepts is:

$$\{ \ \emptyset, \{3\}, \{6\}, \{3, 6\}, \{5, 6\} \ \}.$$

The corresponding family of concepts is:

$$\begin{aligned} \{ \ & (\emptyset, \{a, b, c, d, e, f, g, h, i\}), (\{3\}, \{a, b, c, g, h\}), (\{6\}, \{a, b, c, d, f\}), \\ & (\{3, 6\}, \{a, b, c\}), (\{5, 6\}, \{a, b, d, f\}) \ \}. \end{aligned}$$

Their supremum is $(\{1, 2, 3, 5, 6\}, \{a, b\})$. The lower approximation is $\underline{lapr}(A) = \{1, 2, 3, 5, 6\}$, which is indeed a superset of A. The family of supersets of A that are extensions of concepts is:

$$\{\{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}.$$

The corresponding family of concepts is:

$$\{ \ (\{1, 2, 3, 5, 6\}, \{a, b\}), (\{1, 2, 3, 4, 5, 6, 7, 8\}, \{a\}) \ \}.$$

Their infimum is $(\{1, 2, 3, 5, 6\}, \{a, b\})$. The upper approximation is $\overline{lapr}(A) = \{1, 2, 3, 5, 6\}$, which is the smallest concept whose extension contains A. Although A is not an extension of a concept, it has the same lower and upper approximations, in contrast with Theorem 1.

With a finite set of objects and a finite set of properties, we obtain a finite lattice. The meet-irreducible and join-irreducible concepts in a concept lattice can be used as the elementary concepts. A concept in a finite concept lattice can be expressed as a join of a finite number of join-irreducible concepts and can also be expressed as a meet of a finite number of meet-irreducible concepts [1]. The extensions of meet-irreducible and join-irreducible concepts are treated as elementary definable sets of objects. Approximation operators can therefore be defined based on those elementary definable subsets.

The meet-irreducible and join-irreducible concepts can be defined as follows [1].

Definition 9. *In a concept lattice L, a concept $(A, B) \in L$ is called join-irreducible if and only if for all $(X_1, Y_1), (X_2, Y_2) \in L$, $(A, B) = (X_1, Y_1) \vee (X_2, Y_2)$ implies $(A, B) = (X_1, Y_1)$ or $(A, B) = (X_2, Y_2)$. The dual notion is called meet-irreducible for a concept $(A, B) \in L$ if and only if for all $(X_1, Y_1), (X_2, Y_2) \in L$, $(A, B) = (X_1, Y_1) \wedge (X_2, Y_2)$ implies $(A, B) = (X_1, Y_1)$ or $(A, B) = (X_2, Y_2)$.*

Let $J(L)$ be the set of all join-irreducible concepts and $M(L)$ be the set of all meet-irreducible concepts in L. A concept (A, B) can be expressed by the join of join-irreducible concepts that are the sub-concepts of (A, B) in $J(L)$. That is

$$(A, B) = \bigvee \{(X, Y) \mid (X, Y) \in J(L), (X, Y) \preceq (A, B)\}. \tag{12}$$

A concept (A, B) can also be expressed by the meet of meet-irreducible concepts that are the super-concepts of (A, B) in $M(L)$. That is

$$(A, B) = \bigwedge \{(X, Y) \mid (X, Y) \in M(L), (A, B) \preceq (X, Y)\}. \tag{13}$$

The lower and upper approximations of a set of objects can be defined based on the extensions of join-irreducible and meet-irreducible concepts [8].

Definition 10. *For a set of objects $A \subseteq U$, its lower and upper approximations are defined by:*

$$\begin{aligned} \underline{lapr}(A) &= extent(\bigvee \{(X, Y) \mid (X, Y) \in J(L), X \subseteq A\}) \\ &= (\bigcup \{X \mid (X, Y) \in J(L), X \subseteq A\})^{**}, \\ \overline{lapr}(A) &= extent(\bigwedge \{(X, Y) \mid (X, Y) \in M(L), A \subseteq X\}) \\ &= \bigcap \{X \mid (X, Y) \in M(L), A \subseteq X\}. \end{aligned} \tag{14}$$

Ganter and Wille have shown that a formal concept in a concept lattice can be expressed by the join of object concepts in which the object is included in the extension of the formal concept [6]. That is, for a formal concept (A, B),

$$(A, B) = \bigvee \{(\{x\}^{**}, \{x\}^{*}) \mid x \in A\}.$$

A formal concept can also be expressed by the meet of property concepts in which the property is included in the intension of the formal concept [6]. That is, for a formal concept (A, B),

$$(A, B) = \bigwedge \{(\{y\}^{*}, \{y\}^{**}) \mid y \in B\}.$$

Therefore, the lower and upper approximations of a set of objects can be defined based on the extensions of object and property concepts.

Definition 11. *For a set of objects $A \subseteq U$, its lower and upper approximations are defined by object and property concepts:*

$$
\begin{aligned}
\underline{lapr}(A) &= extent(\bigvee\{(\{x\}^{**}, \{x\}^{*}) \mid x \in U, \{x\}^{**} \subseteq A\}),\\
&= (\bigcup\{\{x\}^{**} \mid x \in U, x \in A\})^{**},\\
\overline{lapr}(A) &= extent(\bigwedge\{(\{y\}^{*}, \{y\}^{**}) \mid y \in V, A \subseteq \{y\}^{*}\}),\\
&= \bigcap\{\{y\}^{*} \mid y \in V, A \subseteq \{y\}^{*}\}. \qquad (15)
\end{aligned}
$$

In fact, this definition can be considered as the extension of granule based definition of rough set approximations in Definition 2.

These definitions of lower and upper approximations are the same as the ones defined in Definition 8. They are regarded as equivalent definitions with slightly different interpretations.

Example 3. In the concept lattice in Figure 1, consider the same set of objects $A = \{3, 5, 6\}$ in Example 2. The family of join-irreducible concepts is:

$$
\begin{array}{ll}
(\{1,2,3\}, \{a,b,g\}), & (\{2,3\}, \{a,b,g,h\}),\\
(\{3\}, \{a,b,c,g,h\}), & (\{4\}, \{a,c,g,h,i\}),\\
(\{5,6\}, \{a,b,d,f\}), & (\{6\}, \{a,b,c,d,f\}),\\
(\{7\}, \{a,c,d,e\}), & (\{6,8\}, \{a,c,d,f\}),\\
(\emptyset, \{a,b,c,d,e,f,g,h,i\}). &
\end{array}
$$

The join-irreducible concepts whose extensions are subsets of A are:

$$
\begin{array}{ll}
(\{3\}, \{a,b,c,g,h\}), & (\{5,6\}, \{a,b,d,f\}),\\
(\{6\}, \{a,b,c,d,f\}), & (\emptyset, \{a,b,c,d,e,f,g,h,i\}).
\end{array}
$$

Thus, according to the Definition 10, the lower approximation is

$$
\begin{aligned}
\underline{lapr}(A) = extent((&\{3\}, \{a,b,c,g,h\}) \bigvee (\{5,6\}, \{a,b,d,f\})\\
&\bigvee (\{6\}, \{a,b,c,d,f\}) \bigvee (\emptyset, \{a,b,c,d,e,f,g,h,i\}))\\
= \{1,2,3,5,6\}&.
\end{aligned}
$$

The family of meet-irreducible concepts is:

$$
\begin{array}{ll}
(\{1,2,3,4,5,6,7,8\}, \{a\}), & (\{1,2,3,5,6\}, \{a,b\}),\\
(\{3,4,6,7,8\}, \{a,c\}), & (\{5,6,7,8\}, \{a,d\}),\\
(\{1,2,3,4\}, \{a,g\}), & (\{5,6,8\}, \{a,d,f\}),\\
(\{2,3,4\}, \{a,g,h\}), & (\{7\}, \{a,c,d,e\}),\\
(\{4\}, \{a,c,g,h,i\}). &
\end{array}
$$

The meet-irreducible concepts whose extensions are supersets of A are:

$$
(\{1,2,3,4,5,6,7,8\}, \{a\}), \quad (\{1,2,3,5,6\}, \{a,b\}).
$$

The upper approximation is

$$\begin{aligned}\overline{lapr}(A) &= extent((\{1,2,3,4,5,6,7,8\},\{a\}) \bigwedge (\{1,2,3,5,6\},\{a,b\})),\\ &= \{1,2,3,5,6\}.\end{aligned}$$

For the set of objects $A = \{3,5,6\}$, its lower approximation equals to its upper approximation. One can see that approximations based on lattice-theoretic operators have some undesirable properties. Other possible formulations are needed.

The upper approximation operator $\overline{lapr}$ is related to the operator *. For any set of objects $A \subseteq U$, we can derive a set of properties A^*. For the set of properties A^*, we can derive another set of objects A^{**}. By property (3), (A^{**}, A^*) is a formal concept. By property (2), we have $A \subseteq A^{**}$. In fact, (A^{**}, A^*) is the smallest formal concept whose extension contains A. That is, for a set of objects $A \subseteq U$, its upper approximation is $\overline{lapr}(A) = A^{**}$.

Thus we can only obtain a weak version of Theorem 1.

Theorem 3. *In a concept lattice $L(U,V,R)$, if A is an extension of a concept, i.e., (A, A^*) is a concept, then $\underline{lapr}(A) = \overline{lapr}(A)$.*

As shown by the examples, the reverse implication in the theorem is not true. This is a limitation of the formulation based on lattice-theoretic operators.

The ideas of approximating a set of objects can be used to define operators that approximate a set of properties. In contract to the approximations of a set of objects, the lower approximation is defined by using meet, and the upper approximation is defined by using join.

Definition 12. (Lattice-theoretic definition) *For a set of properties $B \subseteq V$, its lower and upper approximations are defined by:*

$$\begin{aligned}\underline{lapr}(B) &= intent(\bigwedge\{(X,Y) \mid (X,Y) \in L, Y \subseteq B\})\\ &= (\bigcup\{Y \mid (X,Y) \in L, Y \subseteq B\})^{**},\\ \overline{lapr}(B) &= intent(\bigvee\{(X,Y) \mid (X,Y) \in L, B \subseteq Y\})\\ &= \bigcap\{Y \mid (X,Y) \in L, B \subseteq Y\}. \end{aligned} \tag{16}$$

Definition 13. *For a set of properties $B \subseteq V$, its lower and upper approximations based on the sets of join-irreducible and meet-irreducible concepts are defined by:*

$$\begin{aligned}\underline{lapr}(B) &= intent(\bigwedge\{(X,Y) \mid (X,Y) \in M(L), Y \subseteq B\})\\ &= (\bigcup\{Y \mid (X,Y) \in M(L), Y \subseteq B\})^{**},\\ \overline{lapr}(B) &= intent(\bigvee\{(X,Y) \mid (X,Y) \in J(L), B \subseteq Y\})\\ &= \bigcap\{Y \mid (X,Y) \in J(L), B \subseteq Y\}. \end{aligned} \tag{17}$$

Definition 14. *For a set of properties $B \subseteq V$, its lower and upper approximations are defined by object and property concepts:*

$$\begin{aligned}\underline{lapr}(B) &= intent(\bigwedge\{(\{y\}^*, \{y\}^{**}) \mid y \in V, \{y\}^{**} \subseteq B\}),\\ &= (\bigcup\{\{y\}^{**} \mid y \in V, y \in B\})^{**},\\ \overline{lapr}(B) &= intent(\bigvee\{(\{x\}^{**}, \{x\}^*) \mid x \in U, B \subseteq \{x\}^*\}),\\ &= \bigcap\{\{x\}^* \mid x \in U, B \subseteq \{x\}^*\}. \end{aligned} \tag{18}$$

The lower approximation of a set of properties B is the intension of the formal concept $((\underline{lapr}(B))^*, \underline{lapr}(B))$, and the upper approximation is the intension of the formal concept $((\overline{lapr}(B))^*, \overline{lapr}(B))$.

4.2 Approximations based on set-theoretic operators

By comparing with the standard rough set approximations, one can observe two problems of the approximation operators defined by using lattice-theoretic operators. The lower approximation of a set of objects A is not necessarily a subset of A. Although a set of objects A is undefinable, i.e., A is not the extension of a formal concept, its lower and upper approximations may be the same. In order to avoid these shortcomings, we present another formulation by using set-theoretic operators.

The extension of a formal concept is a definable set of objects. A system of definable sets can be derived from a concept lattice.

Definition 15. *For a formal concept lattice L, the family of all extensions is given by:*

$$EXT(L) = \{extent(X, Y) \mid (X, Y) \in L\}. \tag{19}$$

The system $EXT(L)$ contains the entire set U and is closed under intersection. Thus, $EXT(L)$ is a closure system [3]. Although one can define the upper approximation by extending Definition 1, one cannot define the lower approximation similarly. Nevertheless, one can still keep the intuitive interpretations of lower and upper approximations. That is, the lower approximation is a maximal set in $EXT(L)$ that are subsets of A, and the upper approximation is a minimal set in $EXT(L)$ that are supersets of A. While an upper approximation is unique (e.g., there is a smallest set in $EXT(L)$ containing A), the maximal set contained in A is generally not unique.

Definition 16. **(Set-theoretic definition)** *For a set of objects $A \subseteq U$, its upper approximation is defined by:*

$$\overline{sapr}(A) = \bigcap\{X \mid X \in EXT(L), A \subseteq X\}, \tag{20}$$

and its lower approximation is a family of sets:

$$\begin{aligned}\underline{sapr}(A) = \{X \mid\ & X \in EXT(L), X \subseteq A,\\ & \forall X' \in EXT(L)(X \subset X' \Longrightarrow X' \not\subseteq A)\}.\end{aligned} \tag{21}$$

The upper approximation $\overline{sapr}(A)$ is the same as $\overline{lapr}(A)$, namely, $\overline{sapr}(A) = \overline{lapr}(A)$. However, the lower approximation is different. An important feature is that a set can be approximated from below by several definable sets of objects. In general, for $A' \in \underline{sapr}(A)$, we have $A' \subseteq \underline{lapr}(A)$.

Example 4. In the concept lattice L of Figure 1, the family of all extensions $EXT(L)$ are:

$$\begin{aligned} EXT(L) = \{ & \emptyset, \{3\}, \{4\}, \{6\}, \{7\}, \\ & \{2,3\}, \{3,4\}, \{3,6\}, \{5,6\}, \{6,8\}, \\ & \{1,2,3\}, \{2,3,4\}, \{6,7,8\}, \{5,6,8\}, \\ & \{1,2,3,4\}, \{5,6,7,8\}, \\ & \{1,2,3,5,6\}, \{3,4,6,7,8\}, \\ & \{1,2,3,4,5,6,7,8\} \}. \end{aligned}$$

For a set of objects $A = \{3,5,6\}$, the lower approximation is given by $\underline{sapr}(A) = \{\{3,6\},\{5,6\}\}$, which is a family of sets of objects. The upper approximation is given by $\overline{sapr}(A) = \{1,2,3,5,6\}$, which is a unique set of objects.

Since a concept in a finite concept lattice can be expressed as a meet of a finite number of meet-irreducible concepts, the family of extensions of meet-irreducible concepts can be used to generate the extensions of all concepts in a finite concept lattice by simply using set intersection. Hence, one can use the family of the extensions of all meet-irreducible concepts to replace the system of the extensions of all concepts in the concept lattice.

Let $EXT(M(L))$ denote the family of extensions of all the meet-irreducible concepts. $EXT(M(L))$ is a subset of $EXT(L)$. The extensions of concepts in the system $EXT(M(L))$ are treated as elementary definable sets of objects. Therefore, the upper approximation of a set of objects is the intersection of extensions in $EXT(M(L))$ that are supersets of the set.

Definition 17. *For a set of objects $A \subseteq U$, its upper approximation is defined by:*

$$\overline{sapr}(A) = \bigcap \{X \mid X \in EXT(M(L)), A \subseteq X\}. \tag{22}$$

This definition of upper approximation is the same as Definition 16. They are equivalent but in different forms.

The lower approximation of a set of objects cannot be defined based on the system $EXT(M(L))$. The meet of some meet-irreducible concepts, whose extensions are subsets of a set of objects, is not necessarily the largest set that is contained in the set of objects.

With respect to property (iii), we have:

$$\text{(ix).} \quad A' \subseteq A \subseteq \overline{sapr}(A), \ \text{ for all } A' \in \underline{sapr}(A).$$

That is, A lies within any of its lower approximation and upper approximation. For the set-theoretic formulation, we have a theorem corresponding to Theorem 1.

Theorem 4. *In a concept lattice $L(U, V, R)$, for a subset of the universe of objects $A \subseteq U$, $\overline{sapr}(A) = A$ and $\underline{sapr}(A) = \{A\}$, if and only if A is an extension of a concept.*

In the new formulation, we resolve the difficulties with the approximation operators $\underline{lapr}$ and $\overline{lapr}$. The lower approximation offers more insights into the notion of approximations. In some situations, the union of a family of definable sets is not necessarily a definable set. It may not be reasonable to insist on a unique approximation. The approximation of a set by a family of sets may provide a better characterization of the set.

5 Related Works

In this section, we provide a review of the existing studies on the comparisons and combinations of rough set analysis and formal concept analysis and their relevance to the present study.

5.1 A brief review of existing studies

Broadly, we can classify existing studies into three groups. The first group may be labeled as the comparative studies [5, 7, 9, 10, 15, 21, 23, 24, 33]. They deal with the comparison of the two approaches with an objective to produce a more generalized data analysis framework. The second group concerns the applications of the notions and ideas of formal concept analysis into rough set analysis [5, 23, 34]. Reversely, the third group focuses on applying concepts and methods of rough set analysis into formal concept analysis [4, 8, 11, 16, 17, 20, 23, 34, 39]. Those studies lead to different types of abstract operators, concept lattices and approximations.

Comparative studies

Kent examined the correspondence between similar notions used in both theories, and argued that they are in fact parallel to each other in terms of basic notions, issues and methodologies [9]. A framework of rough concept analysis was introduced as a synthesis of the two theories. Based on this framework, Ho developed a method of acquiring rough concepts [7], and Wu, Liu and Li proposed an approach for computing accuracies of rough concepts and studied the relationships between the indiscernibility relations and accuracies of rough concepts [27].

The notion of a formal context has been used in many studies under different names. Shafer used a compatibility relation to interpret the theory of evidence [18, 19]. A compatibility relation is a binary relation between two universes, which is in fact a formal context. Wong, Wang and Yao investigated approximation operators over two universes with respect to a compatibility relation [25, 26]. Düntsch and Gediga referred to those operators as modal-style operators and studied a class of such operators in data analysis [5]. The derivation operator in formal concept analysis is a sufficiency operator, and the rough set approximation operators are the necessity and possibility operators used in modal logics. By focusing on modal-style operators, we have a unified operator-oriented framework for the study of the two theories.

Pagliani used a Heyting algebra structure to connect concept lattices and approximation spaces together [10]. Based on the algebra structure, concept lattices and approximation spaces can be transformed into each other. Wasilewski demonstrated that formal contexts and general approximation spaces can be mutually represented [21]. Consequently, rough set analysis and formal concept analysis can be viewed as two related and complementary approaches for data analysis. It is shown that the extension of a formal concept is a definable set in the approximation space. Qi *et al.* argued that two theories have much in common in terms of the goals and methodologies [15]. They emphasized the basic connection and transformation between a concept lattice and a partition.

Wolski investigated Galois connections in formal concept analysis and their relations to rough set analysis [23]. A logic, called S4.t, is proposed as a good tool for approximate reasoning to reflect the formal connections between formal concept analysis and rough set analysis [24].

Yao compared the two theories based on the notions of definability, and showed that they deal with two different types of definability [33]. Rough set analysis studies concepts that are defined by disjunctions of properties. Formal concept analysis considers concepts that are definable by conjunctions of properties.

Based on those comparative studies, one can easily adopt ideas from one theory to another. The applications of rough set always lead to approximations and reductions in formal concept analysis. The approximations of formal concept analysis result in new types of concepts and concept lattices.

Approximations and reductions in concept lattices

Many studies considered rough set approximations in formal concept lattice [4, 8, 11, 16, 17, 20, 23]. They will be discussed in Section 5.2. The present study is in fact a continuation in the same direction.

Zhang, Wei and Qi examined property (object) reduction in concept lattice using the ideas from rough set analysis [39]. The minimal sets of properties (objects) are determined based on criterions that the reduced lattice and the original lattice show certain common features or structures. For example, two lattices are isomorphic.

Concept lattices in rough sets

Based on approximation operators, one can construct additional concept lattices. Those lattices, their properties, and connections to the original concept lattice are studied extensively by Düntsch and Gediga [5], and Wolski [23]. The results provide more insights into data analysis using modal-style operators.

Yao examined semantic interpretations of various concept lattices [34]. One can obtain different types of inference rules regarding objects and properties. To reflect their physical meanings, the notions of object-oriented and property-oriented concept lattices are introduced.

5.2 Approximations in formal concept lattice

Saquer and Deogun suggested that all concepts in a concept lattice can be considered as definable, and a set of objects can be approximated by concepts whose extensions approximate the set of objects [16, 17]. A set of properties can be similarly approximated by using intensions of formal concepts.

For a given set of objects, it may be approximated by extensions of formal concepts in two steps. The classical rough set approximations for a given set of objects are first computed. Since the lower and upper approximations of the set are not necessarily the extensions of formal concepts, they are then approximated again by using derivation operators of formal concept analysis.

At the first step, for a set of objects $A \subseteq U$, the standard lower approximation $\underline{apr}(A)$ and upper approximation $\overline{apr}(A)$ are obtained. At the second step, the lower approximation of the set of objects A is defined by the extension of the formal concept $(\underline{apr}(A)^{**}, \underline{apr}(A)^{*})$. The upper approximation of the set of objects A is defined by the extension of the formal concept $(\overline{apr}(A)^{**}, \overline{apr}(A)^{*})$. That is,

$$\begin{aligned} \underline{eapr}(A) &= \underline{apr}(A)^{**}, \\ \overline{eapr}(A) &= \overline{apr}(A)^{**}. \end{aligned}$$

If $\underline{apr}(A) = \overline{apr}(A)$, we have $\underline{apr}(A)^{**} = \overline{apr}(A)^{**}$. Namely, for a definable set A, its lower and upper formal concept approximations are the same. However, the reverse implication is not true. A set of objects that has the same lower and upper approximations may not necessarily be a definable set. This shortcoming of their definition is the same as the lattice-theoretic formulations of approximations.

Hu *et al.* suggested an alternative formulation [8]. Instead of defining an equivalence relation, they defined a partial order on the universe of objects. For an object, its principal filter, which is the set of objects "greater than or equal to" the object and is called the partial class by Hu *et al.*, is the extension of a formal concept. The family of all principal filters is the set of join-irreducible elements of the concept lattice. Similarly, a partial order relation can be defined on the set of properties. The family of meet-irreducible elements of the concept lattice can be constructed. The lower and upper approximations can be defined based on the families of meet- and join-irreducible elements in concept lattice. Their definitions are similar to our lattice-theoretic definitions. However, their definition of lower approximation has the same shortcoming of Saquer and Deogun's definition [16].

Some researchers used two different systems of concepts to approximate a set of objects or a set of properties [4, 11, 14, 20, 23]. In addition to the derivation operator, one can define the two rough set approximation operators [5, 23, 25, 26, 34].

$$\begin{aligned} A^{\Box} &= \{y \in V \mid Ry \subseteq A\}, \\ A^{\Diamond} &= \{y \in V \mid Ry \cap A \neq \emptyset\}, \end{aligned}$$

and

$$\begin{aligned} B^{\Box} &= \{x \in U \mid xR \subseteq B\}, \\ B^{\Diamond} &= \{x \in U \mid xR \cap B \neq \emptyset\}. \end{aligned}$$

Düntsch and Gediga referred to *, $^\Box$ and $^\Diamond$ as modal-style operators, called sufficiency operator, necessity operator and possibility operator, respectively [4, 5].

The two operators can be used to define two different types of concepts and concept lattices [5, 34]. A pair (A, B) is called an object-oriented concept if $A = B^\Diamond$ and $B = A^\Box$. The family of all object-oriented concepts forms a complete lattice, denoted as $L_o(U, V, R)$. A pair (A, B) is called a property-oriented concept if $A = B^\Box$ and $B = A^\Diamond$. The family of all property-oriented concepts also forms a complete lattice, denoted as $L_p(U, V, R)$. Similar to the formal concept lattice, the set of objects A is referred to as the extension of the concept, and the set of properties B is referred to as the intension of the concept. With respect to those new concept lattices, one can apply the formulation of approximations discussed previously in a similar way. For example, we may study the approximations of a set of objects by using an object-oriented concept lattice.

Another class of approximation operators can be derived by the combination of operators $^\Box$ and $^\Diamond$. The combined operators $^{\Box\Diamond}$ and $^{\Diamond\Box}$ have following important properties [4]:

$$
\begin{aligned}
&1).\ ^{\Box\Diamond} \text{ is a closure operator on } U \text{ and } V,\\
&2).\ ^{\Box\Diamond} \text{ and } ^{\Diamond\Box} \text{ are dual to each other},\\
&3).\ ^{\Diamond\Box} \text{ is an interior operator on } U \text{ and } V.
\end{aligned}
$$

Based on those properties, approximation operators can be defined [4, 11, 20, 23]. The lower and upper approximations of a set of objects and a set of properties can be defined, respectively, based on two systems:

$$\underline{rapr}(A) = A^{\Box\Diamond},\ \underline{rapr}(B) = B^{\Box\Diamond},$$

and

$$\overline{rapr}(A) = A^{\Diamond\Box},\ \overline{rapr}(B) = B^{\Diamond\Box}.$$

The operators $^{\Box\Diamond}$ and $^{\Diamond\Box}$ and the corresponding rough set approximations have been used and studied by many authors, for example, Düntsch and Gediga [4], Pagliani [10], Pagliani and Chakraborty [11], Pei and Xu [14], Shao and Zhang [20], and Wolski [23, 24].

If a set of objects A equals to its lower approximation $\underline{rapr}(A)$, we say that A is a definable set of objects in the system $L_o(U, V, R)$. If the set of objects A equals to its upper approximation $\overline{rapr}(A)$, we say that A is a definable set of objects in the system $L_p(U, V, R)$. The lower and upper approximations of a set of objects are equal if and only if the set of objects is a definable set in both systems $L_o(U, V, R)$ and $L_p(U, V, R)$. Similarly, the lower and upper approximations of a set of properties are equal if and only if the set of properties is a definable set in both systems $L_o(U, V, R)$ and $L_p(U, V, R)$.

6 Conclusion

An important issue of rough set analysis is the approximations of undefinable sets using definable sets. In the classical rough set theory, the family of definable sets is a subsystem of the power set of a universe. There are many approaches to construct subsystems of definable sets [30, 36]. Formal concept analysis provides an approach for the construction of a family of definable sets. It represents a different type of definability. The notion of approximations can be introduced naturally into formal concept analysis.

Formal concepts in a formal concept lattice correspond to definable sets. Two types of approximation operators are investigated. One is based on the lattice-theoretic formulation and the other is based on the set-theoretic formulation. Their properties are studied in comparison with the properties of classical rough set approximation operators. A distinguishing feature of the lower approximation defined by set-theoretic formulation is that a subset of the universe is approximated from below by a family of definable sets, instead of a unique set in the classical rough set theory.

The theory of rough sets and formal concept analysis capture different aspects of data. They can represent different types of knowledge embedded in data sets. The introduction of the notion of approximations into formal concept analysis combines the two theories. It describes a particular characteristic of data, improves our understanding of data, and produces new tools for data analysis.

The sufficiency operators * is an example of modal-style operators [4, 5, 33]. One can study the notion of rough set approximations in a general framework in which various modal-style operators are defined [4, 5, 10, 33].

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